Question 1

(a) Prove that if A, B are commuting square matrices (i.e., matrices which satisfy $AB = BA$) then $\exp(A+B) = \exp(A)\exp(B)$.

Solution.

$$\begin{aligned}
\exp(A+B) &= \sum_{n=0}^{\infty} \frac{1}{n!}(A+B)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} A^k B^{n-k} \qquad = [\text{now denoting } j = n-k] \\
&= \sum_{n=0}^{\infty} \sum_{\substack{j,k\geq 0 \\ j+k=n}} \frac{1}{k!\,j!} A^k B^j = \sum_{k=0}^{\infty} \frac{1}{k!} A^k \cdot \sum_{j=0}^{\infty} \frac{1}{j!} B^j = e^A e^B.
\end{aligned}$$

(b) Compute e^{tA} for the following matrices A:

i. $A = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}$

Solution. Diagonalizing A, we find that it has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 3$, with corresponding eigenvectors $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Equivalently, this means that A can be expressed as $A = PDP^{-1}$, where

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}.$$

Therefore, using standard properties of matrix exponentials, we have

$$\begin{aligned}
\exp(tA) = Pe^{tD}P^{-1} &= \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \\
&= \begin{pmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 3e^{2t} - 2e^{3t} & -e^{2t} + e^{3t} \\ 6e^{2t} - 6e^{3t} & -2e^{3t} + 3e^{3t} \end{pmatrix}.
\end{aligned}$$

ii. $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Solution. Observe that $A^2 = -I$. We can therefore write the power series expansion as

$$\begin{aligned} e^{tA} &= I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \frac{t^4}{4!}A^4 + \ldots \\ &= I + tA + \frac{t^2}{2!}(-I) + \frac{t^3}{3!}(-A) + \frac{t^4}{4!}I + \frac{t^5}{5!}A + \ldots \\ &= I\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots\right) + A\left(1 - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots\right) \\ &= \cos t \cdot I + \sin t \cdot A = \begin{pmatrix} \cos t & 0 \\ 0 & \cos t \end{pmatrix} + \begin{pmatrix} 0 & \sin t \\ -\sin t & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \end{aligned}$$

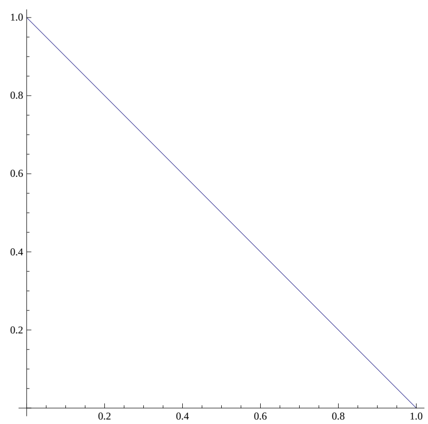
Question 2

Let $T : [0,1] \to [0,1]$ be the interval map defined by

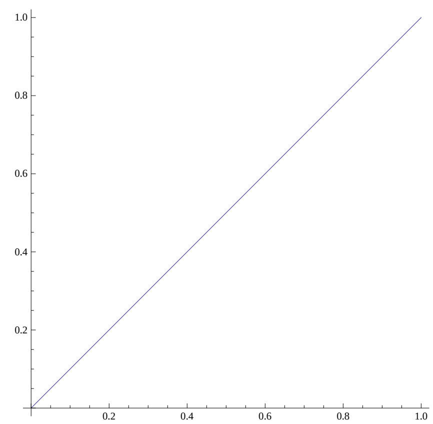
$$T(x) = 1 - x.$$

(a) Sketch the graph of T and of its second iteration $T \circ T$.

Solution. $(T \circ T)(x) = T(T(x)) = 1 - (1 - x) = x$ (the identity function), so the graphs are as follows:



The graph of T



The graph of $T \circ T$

(b) Identify all the fixed points of T and all the k-cycles for $k \geq 1$.

Solution. The fixed points of T are solutions of the equation $x = 1 - x$, which gives a unique fixed point $x = 1/2$. Next, to find the 2-cycles, we solve the equation $T \circ T(x) = x$. Since $T \circ T(x) \equiv x$, the solutions are all the numbers in $[0,1]$. So any $x \in [0,1]$, except $x = 1/2$ (which is a fixed point, i.e., a 1-cycle) belongs to a 2-cycle $(x, T(x)) = (x, 1 - x)$.

Finally, observe that there are no k-cycles for any $k > 2$; the 1-cycles and 2-cycles already cover all the numbers in $[0,1]$, so there are no remaining points to belong to a k-cycle for higher k. An alternative explanation is that since $T \circ T$ is the identity function id, it follows that all subsequent iterates T^k of the map are either T itself (for odd k) or the identity function (for even k):

$$\begin{aligned} T^3 &= (T^2) \circ T = \mathrm{id} \circ T = T, \\ T^4 &= \mathrm{id} \circ \mathrm{id} = \mathrm{id}, \\ T^5 &= \mathrm{id} \circ \mathrm{id} \circ T = T, \\ &\ \ \vdots \end{aligned}$$

As a result, when we try to solve the equation $T^k(x) = x$ to find candidate members of a k-cycle, we will find the same solutions as for T (for odd k) or for id (for even k).