Solutions to practice questions for the final

1. You are given the linear system of equations

$$\begin{cases} 2x_1 + 4x_2 + x_3 + x_4 = 8 \\ x_1 + 2x_2 + x_3 = 5 \\ -x_1 - 2x_2 + x_3 - 2x_4 = -1 \\ x_1 + 2x_2 + x_4 = 3 \end{cases}$$

(a) Write an augmented matrix representing the system.

Solution.

$$\left(\begin{array}{rrrr|r} 2 & 4 & 1 & 1 & 8 \\ 1 & 2 & 1 & 0 & 5 \\ -1 & -2 & 1 & -2 & -1 \\ 1 & 2 & 0 & 1 & 3 \end{array}\right)$$

(b) Find a reduced row echelon form (RREF) matrix that is row-equivalent to the augmented matrix.

Solution.

$$\left(\begin{array}{rrrr|r} 1 & 2 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(c) Find the general solution of the system.

Solution.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

(d) Write the homogeneous system of equations associated with the above (nonhomogeneous) system and find its general solution.

Solution. The homogeneous system is represented by the augmented matrix

$$\left(\begin{array}{cccc|c} 2 & 4 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ -1 & -2 & 1 & -2 & 0 \\ 1 & 2 & 0 & 1 & 0 \end{array}\right)$$

This is the same as the original system except that the rightmost column is the zero vector. The equivalent RREF is therefore

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

and the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \lambda_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

2. Define $M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & -1 \end{pmatrix}$.

(a) Find the inverse matrix of M using elementary row operations.

Solution.

$$M^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & -1 \end{pmatrix}$$

(b) Compute the adjoint matrix $\mathrm{adj}(M)$ using the definition of the adjoint matrix.

(c) Compute $\det(M)$ and verify using the results of the above computations that $\mathrm{adj}(M) = \det(M)M^{-1}$.

Solution. $\det(M) = 1 \cdot 1 \cdot (-1) = -1$, since M is a lower-triangular matrix.

3. Define $M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$.

(a) Name an easily observable property of the matrix M that guarantees that it is diagonalizable.

Solution. M is a symmetric matrix.

(b) Compute the characteristic polynomial $P_M(\lambda)$ of M and find all its zeros.

Solution.

$$P_M(\lambda) = \det(\lambda I - M) = \det \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda - 3 & 1 \\ 0 & 1 & \lambda - 3 \end{pmatrix} = \lambda[(\lambda - 3)^2 - 1]$$
$$= \lambda(\lambda^2 - 6\lambda + 8) = \lambda(\lambda - 2)(\lambda - 4)$$

The zeros of $P_M(\lambda)$ (which are the eigenvalues of M) are $\lambda_1 = 0, \lambda_2 = 2$ and $\lambda_3 = 4$.

(c) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of M.

Solution. A basis of eigenvectors corresponding to the eigenvalues λ_1, λ_2 and λ_3 is given by

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

(d) Find a 3×3 invertible matrix P and a 3×3 diagonal matrix D such that $M = PDP^{-1}$.

Solution. The matrix P is constructed by putting the vectors v_1, v_2, v_3 in the columns of a matrix and the matrix D is the diagonal matrix having the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ in the diagonal (they need to appear in the same order as the order of the eigenvectors in the columns of P). Thus:

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

4. (a) Let $\{u, v\}$ be an orthonormal basis for $\mathbb{R}^2$. Let a, b be two real numbers such that $a^2 + b^2 = 1$. Show that the vectors $\{w, z\}$ given by

$$\begin{aligned} w &= au + bv \\ z &= -bu + av \end{aligned}$$

also form an orthonormal basis for $\mathbb{R}^2$.

Solution. Compute the dot products $w \cdot w$, $w \cdot z$ and $z \cdot z$, using the information that $u \cdot u = v \cdot v = 1$ and $u \cdot v = 0$:

$$\begin{aligned} w \cdot w &= (au + bv) \cdot (au + bv) \\ &= a^2(u \cdot u) + ab(u \cdot v) + ab(v \cdot u) + b^2(v \cdot v) = a^2 + b^2 = 1, \\ w \cdot z &= (au + bv) \cdot (-bu + av) \\ &= -ab(u \cdot u) + ab(v \cdot v) + a^2(u \cdot v) - b^2(u \cdot v) = -ab + ab = 0, \end{aligned}$$

and similarly $z \cdot z = 1$. So w, z are orthonormal. Note in particular that they are linearly independent (otherwise one is a scalar multiple of the other, in contradiction to the orthogonality), and since $\mathbb{R}^2$ is two-dimensional this implies that they are a basis.

(b) Define the linear subspace $U = \text{span}\{(1, 1, 1)\}$ of $\mathbb{R}^3$. Find a basis for its orthogonal complement $U^\perp$.

Solution. $U^\perp$ consists of all vectors (x, y, z) such that $(x, y, z) \cdot (1, 1, 1) = x + y + z = 0$. This is a "system" of one linear equation in three variables. Its solution set is found (using the standard method) to be

$$\text{span}\{v_1 = (1, -1, 0),\ v_2 = (1, 0, -1)\}.$$

These two vectors are linearly independent so they form a basis for $U^\perp$.

(c) Find an *orthogonal* basis for the space $U^\perp$ defined above.

Solution. Apply the Gram-Schmidt orthogonalization process to the vectors v_1, v_2 to get an orthogonal basis

$$\begin{aligned} u_1 &= v_1 = (1, -1, 0), \\ u_2 &= v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = (1, 0, -1) - \tfrac{1}{2}(1, -1, 0) = (1/2, /2, -1). \end{aligned}$$

5. (a) If u, v are two linearly independent vectors in a vector space V, prove that $u + v, u - v$ are also linearly independent.

Solution. Denote $p = u + v$ and $q = u - v$. If a, b are scalars such that $ap + bq = \underline{0}$, then

$$\underline{0} = a(u + v) + b(u - v) = (a + b)u + (a - b)v.$$

This is a linear combination of u and v that gives the zero vector. Since u, v are linearly independent this implies that the coefficients of the linear combination are both zero. So:

$$a + b = 0, \qquad a - b = 0$$

and this clearly implies (by considering the sum and the difference of the two equations) that $a = b = 0$, proving linear independence.

(b) If $u_1, \dots, u_k$ are vectors that span $\mathbb{R}^5$, what are the possible values of k?

Solution. $\dim(\mathbb{R}^5) = 5$, so a spanning set (which can be diluted to get a basis) must contain at least 5 vectors. Hence $k \geq 5$.

(c) If M is a 5×5 matrix such that the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

form a basis for the kernel $\ker(M)$, find the rank of M.

Solution. The data implies in particular that $\dim(\ker(M)) = 2$ (the nullity of M). By the dimension formula, $\dim(\ker(M)) + \mathrm{rank}(M) = 5$, so $\mathrm{rank}(M) = 3$.