Final exam notes

- ⋆ Final exam date, time and place: March 16, Young Hall 198 at 3:30 PM.
- ⋆ Exam duration: 2 hours.
- ⋆ Exam type: closed book exam — no written material or electronic devices allowed.
- ⋆ Exam material: the exam will cover all the material covered in class, with a greater emphasis on material covered after the second midterm.

Practice questions for the final

1. You are given the linear system of equations

$$\begin{cases} 2x_1 + 4x_2 + x_3 + x_4 = 8 \\ x_1 + 2x_2 + x_3 = 5 \\ -x_1 - 2x_2 + x_3 - 2x_4 = -1 \\ x_1 + 2x_2 + x_4 = 3 \end{cases}$$

(a) Write an augmented matrix representing the system.

(b) Find a reduced row echelon form (RREF) matrix that is row-equivalent to the augmented matrix.

(c) Find the general solution of the system. It should be expressed in the form

$$\{X_0 + \lambda_1 Y_1 + \lambda_2 Y_2 + \ldots + \lambda_k Y_k\}$$

where $X_0, Y_1, \ldots, Y_k$ are some vectors in $\mathbb{R}^4$ and $\lambda_1, \ldots, \lambda_k$ are free parameters.

(d) Write the homogeneous system of equations associated with the above (nonhomogeneous) system and find its general solution.

2. Define $M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & -1 \end{pmatrix}$.

(a) Find the inverse matrix of M using elementary row operations. Note: It is highly recommended to check that the matrix you found actually satisfies $MM^{-1} = I$.

(b) Compute the adjoint matrix $\operatorname{adj}(M)$ using the definition of the adjoint matrix.

(c) Compute $\det(M)$ and verify using the results of the above computations that $\operatorname{adj}(M) = \det(M)M^{-1}$.

3. Define $M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$.

(a) Name an easily observable property of the matrix M that guarantees that it is diagonalizable.

(b) Compute the characteristic polynomial $P_M(\lambda)$ of M and find all its zeros.

(c) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of M.

(d) Find a 3×3 invertible matrix P and a 3×3 diagonal matrix D such that $M = PDP^{-1}$ (there is no need to compute P^{-1} as long as your answer for P is correct).

4. (a) Let $\{u, v\}$ be an orthonormal basis for $\mathbb{R}^2$. Let a, b be two real numbers such that $a^2 + b^2 = 1$. Show that the vectors $\{w, z\}$ given by

$$\begin{aligned} w &= au + bv \\ z &= -bu + av \end{aligned}$$

also form an orthonormal basis for $\mathbb{R}^2$.

(b) Define the linear subspace $U = \operatorname{span}\{(1, 1, 1)\}$ of $\mathbb{R}^3$. Find a basis for its orthogonal complement $U^\perp$.

(c) Find an *orthogonal* basis for the space $U^\perp$ defined above.

5. (a) If u, v are two linearly independent vectors in a vector space V, prove that $u + v, u - v$ are also linearly independent.

(b) If $u_1, \ldots, u_k$ are vectors that span $\mathbb{R}^5$, what are the possible values of k?

(c) If M is a 5×5 matrix such that the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

form a basis for the kernel $\ker(M)$, find the rank of M.