Midterm Exam 1 Solutions

1. Find the general form of the solution of the linear system

$$\begin{cases} 2x - 2y - 4z - 6w = -2 \\ -x + y + 2z + 3w = 1 \\ 5x - 4y + 4z + 4w = 1 \\ 4x - 3y + 6z + 7w = 2 \end{cases}$$

Solution. We encode the system as the augmented matrix

$$\left(\begin{array}{cccc|c} 2 & -2 & -4 & -6 & -2 \\ -1 & 1 & 2 & 3 & 1 \\ 5 & -4 & 4 & 4 & 1 \\ 4 & -3 & 6 & 7 & 2 \end{array}\right)$$

Using the Gaussian elimination algorithm, we can perform a sequence of elementary row operations to bring the matrix to its reduced row echelon form. This results in the equivalent augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 0 & 12 & 16 & 5 \\ 0 & 1 & 14 & 19 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Here z and w are the independent (non-pivot) variables, so we denote $z = s$ and $w = t$ where s, t are parameters taking arbitrary real values. Therefore the general solution to the system is of the form

$$x = 5 - 12s - 16t, \quad y = 6 - 14s - 19t, \quad z = s, \quad w = t,$$

or in vector notation

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -12 \\ -14 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -16 \\ -19 \\ 0 \\ 1 \end{pmatrix}$$

2. For each of the following geometric forms, write an example of a linear system of equations in 3 variables x, y, z for which the solution set is of that form:

(a) the empty set
(b) a single point
(c) a line
(d) a plane

(It is okay to write any of the systems as an augmented matrix.)

Solution.

$$\text{(a)} \qquad 0x + 0y + 0z = 1$$

$$\text{(b)} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\text{(c)} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \end{array}\right)$$

$$\text{(d)} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \end{array}\right)$$

3. (a) If $u, v \in \mathbb{R}^3$ satisfy $||u|| = 3, ||v|| = 8$ and $u \cdot v = 12$, what is the angle between u and v?

Solution. Denote the angle by θ. By the definition of the angle, we have

$$\cos(\theta) = \frac{u \cdot v}{||u||\,||v||} = \frac{12}{3 \times 8} = \tfrac{1}{2}.$$

Therefore $\theta = \frac{\pi}{3} = 60°$.

(b) If $u, v \in \mathbb{R}^n$ are vectors, show that

$$||u + v||^2 + ||u - v||^2 = 2||u||^2 + 2||v||^2.$$

Solution. Use the definition of the square of the length of a vector w as the dot product $w \cdot w$, and the usual properties of the dot product, to write

$$\begin{aligned} ||u + v||^2 + ||u - v||^2 &= (u + v) \cdot (u + v) + (u - v) \cdot (u - v) \\ &= u \cdot (u + v) + v \cdot (u + v) + u \cdot (u - v) - v \cdot (u - v) \\ &= u \cdot u + u \cdot v + v \cdot u + v \cdot v + u \cdot u - u \cdot v - v \cdot u + v \cdot v \\ &= 2u \cdot u + 2v \cdot v = 2||u||^2 + 2||v||^2. \end{aligned}$$

4. Find numbers a, b, c such that $a + b + c = 5$ and the plane

$$\{ax + by + cz = 0\}$$

in $\mathbb{R}^3$ contains the points $(0, 0, 0)$, $(0, 3, -3)$ and $(-4, 1, 1)$.

Solution. The data in the question gives us the equations

$$\left\{\begin{array}{rcrcrcl} a & + & b & + & c & = & 5 \\ 0a & + & 0b & + & 0c & = & 0 \\ 0a & + & 3b & - & 3c & = & 0 \\ -4a & + & b & + & c & = & 0 \end{array}\right.$$

in the unknowns a, b, c. This is easy to solve in the usual way to get the solution

$$a = 1, \quad b = 2, \quad c = 2.$$

5. (a) Compute the matrix products MN, $M^\top M$, $MM^\top$, where

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 4 \\ 0 & 1 \\ 2 & 0 \end{pmatrix}.$$

Solution.

$$MN = \begin{pmatrix} 6 & 6 \\ 2 & 0 \end{pmatrix}, \quad M^\top M = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 10 \end{pmatrix}, \quad MM^\top = \begin{pmatrix} 14 & 3 \\ 3 & 1 \end{pmatrix}.$$

(b) Define what it means for a matrix A to be symmetric.

Solution. A matrix A is symmetric if $A = A^\top$. That is, the number of rows has to equal the number of columns and for each i and j, the entry of A in the i-th row and j-th column is equal to the entry in the j-th row and i-th column.

(c) Show that for any matrix A, the product $AA^\top$ is always a symmetric matrix.

Solution. Define $B = AA^\top$. We verify that $B^\top = B$: indeed, we have

$$B^\top = (AA^\top)^\top = (A^\top)^\top A^\top = AA^\top = B$$

where we use the facts that $(MN)^\top = N^\top M^\top$ and that $(A^\top)^\top = A$.