Midterm Exam 2 Solutions

1. For each of the following matrices, determine if it is invertible, and if it is, find its inverse matrix.

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -2 \\ 1 & 3 & -1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{pmatrix}$

Solution. (a) Recall that the inverse of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ exists if its determinant $ad - bc \neq 0$, and in that case the inverse matrix is equal to $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. In this case $ad - bc = 4 - 6 = -2$ so the inverse matrix exists and is $\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$.

(b) This matrix has a row of zeroes so its determinant is 0 and therefore it is not invertible.

(c) The determinant of this matrix is also 0 so it is not invertible.

(d) By applying elementary row operations to bring the matrix to reduced row echelon form and performing the same operations in parallel on the identity matrix, we get

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right).$$

Since the RREF is the identity matrix, it follows that the matrix is invertible (this could be checked separately by computing the determinant), and the inverse matrix is

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & -2 \\ 0 & -1 & 1 \end{pmatrix}.$$

2. Write the 6 different permutations of $1, 2, 3$. For each permutation, find its sign and its inverse permutation.

Solution.

Permutation	sign	inverse
$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$	+1	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$	-1	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$	-1	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$	+1	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$	+1	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$	-1	$\begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix}$

3. Define the matrices

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3 & -2 & 2 & 1 \\ 0 & 15 & 0 & 1 \\ 5 & 5 & 5 & 5 \end{pmatrix}, \quad N = \begin{pmatrix} 3 & 9 & -1 & -1 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Compute the determinants: $\det(M), \det(N), \det(MN), \det(M^{-1}), \det(\mathrm{adj}(M))$.
Hint: Recall that the adjoint matrix $\mathrm{adj}(M)$ satisfies $M \cdot \mathrm{adj(M)} = \det(M)I$.

Solution.

$$\begin{aligned}
\det(M) &= (-1)\det\begin{pmatrix} 3 & 2 & 1 \\ 0 & 0 & 1 \\ 5 & 5 & 5 \end{pmatrix} = (-1)(-1)\det\begin{pmatrix} 3 & 2 \\ 5 & 5 \end{pmatrix} = 5, \\
\det(N) &= 3 \times 1 \times 2 \times 2 = 12 \qquad \text{(an upper triangular matrix)}, \\
\det(MN) &= \det(M)\det(N) = 5 \times 12 = 60, \\
\det(M^{-1}) &= \frac{1}{\det(M)} = \tfrac{1}{5}, \\
\det(\mathrm{adj}(M)) &= \det\Big((\det(M))M^{-1}\Big) = \det\left(5M^{-1}\right) = 5^4 \det\left(M^{-1}\right) = 5^3 = 125.
\end{aligned}$$

Note: when multiplying an $n \times n$ matrix by a scalar c, the determinant gets multiplied by c a total of n times (one for each row), so the new determinant is c^n times the original determinant:

$$\det(cM) = c^n \det(M)$$

4. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by $L(v) = Av$, where $A = \begin{pmatrix} -15 & -30 \\ 10 & 20 \end{pmatrix}$. Find the two eigenvalues λ_1, λ_2 of A and for each eigenvector find an associated eigenvector.

Solution. $\det(\lambda I - A) = \det \begin{pmatrix} \lambda + 15 & 30 \\ -10 & \lambda - 20 \end{pmatrix} c = \lambda^2 - 5\lambda = \lambda(\lambda - 5)$. This is 0 iff $\lambda = 0$ or 5. Thus $\lambda_1 = 0, \lambda_2 = 5$ are the two eigenvalues. Solving the equations

$$(\lambda_1 I - A)v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad (\lambda_2 I - A)v_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

in the usual way leads to the corresponding eigenvectors

$$v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \qquad v_1 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$

5. Let M be a 5×5 matrix. Let N be the matrix obtained by reversing the order of the rows of M.

(a) Are M and N row-equivalent? If so, describe a sequence of elementary row operations that transforms M into N.

Solution. Yes, they are row-equivalent. The operations are: swap rows 1 and 5; swap rows 2 and 4.

(b) Find a matrix A such that $N = AM$.

Solution. We can express the row operations as multiplication from the left by the corresponding elementary matrices: $N = E_{R_1 \leftrightarrow R_5} E_{R_2 \leftrightarrow R_4} M$. Therefore the matrix A is

$$E_{R_1 \leftrightarrow R_5} E_{R_2 \leftrightarrow R_4} = E_{R_1 \leftrightarrow R_5} E_{R_2 \leftrightarrow R_4} I$$

which is the matrix obtained from I by performing the same operations. That is,

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(c) If $\det(M) = 3$, find $\det(N)$.

Solution.

$$\begin{aligned}\det(N) &= \det(AM) = \det(A)\det(M) = \det(E_{R_1 \leftrightarrow R_5})\det(E_{R_2 \leftrightarrow R_4})\det(M) \\ &= (-1) \times (-1) \times 3 = 3\end{aligned}$$