Solutions to practice question set 1

1. Find the general form of the solution of the linear system

$$\begin{cases} & & & & x_3 & - & 2x_4 & = & -2 \\ x_1 & + & 4x_2 & + & x_3 & + & 7x_4 & = & 1 \\ 2x_1 & + & 8x_2 & + & 3x_3 & + & 12x_4 & = & 0 \\ x_1 & + & 4x_2 & - & x_3 & + & 11x_4 & = & 5 \end{cases}$$

Solution. First, write the system as an augmented matrix

$$\left(\begin{array}{cccc|c} 0 & 0 & 1 & -2 & -2 \\ 1 & 4 & 1 & 7 & 1 \\ 2 & 8 & 3 & 12 & 0 \\ 1 & 4 & -1 & 11 & 5 \end{array}\right)$$

Next, perform the Gaussian elimination procedure to bring the augmented matrix to reduced row echelon form (RREF) by applying a sequence of elementary row operations:

$$\left(\begin{array}{cccc|c} 0 & 0 & 1 & -2 & -2 \\ 1 & 4 & 1 & 7 & 1 \\ 2 & 8 & 3 & 12 & 0 \\ 1 & 4 & -1 & 11 & 5 \end{array}\right) \xrightarrow{\text{swap } R_1,R_2} \left(\begin{array}{cccc|c} 1 & 4 & 1 & 7 & 1 \\ 0 & 0 & 1 & -2 & -2 \\ 2 & 8 & 3 & 12 & 0 \\ 1 & 4 & -1 & 11 & 5 \end{array}\right)$$

$$\xrightarrow[R_4-R_1]{R_3-2R_1} \left(\begin{array}{cccc|c} 1 & 4 & 1 & 7 & 1 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & -2 & 4 & 4 \end{array}\right) \xrightarrow[\substack{R_3-R_2 \\ R_4+2R_2}]{R_1-R_2} \left(\begin{array}{cccc|c} 1 & 4 & 0 & 9 & 3 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Translating this back into a system of equations, we have reached the equivalent system

$$\begin{cases} x_1 & + & 4x_2 & & & + & 9x_4 & = & 3 \\ & & & & x_3 & - & 2x_4 & = & -2 \end{cases}$$

Here we have two independent (non-pivot) variables x_2 and x_4, so the solution can be written as

$$x_2 = s, \quad x_4 = t, \quad x_1 = 3 - 4s - 9t, \quad x_3 = -2 + 2t,$$

where s, t are parameters taking arbitrary real values. In vector notation this translates to

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 - 4s - 9t \\ s \\ -2 + 2t \\ t \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -2 \\ 0 \end{pmatrix} + s \begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -9 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

(the recommended form for the final answer).

2. For each of the following matrices, find an augmented matrix in Reduced Row Echelon Form (RREF) that is row-equivalent to it:

(a) $\left(\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 0 \\ 0 & 1 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 \end{array}\right)$ (b) $\left(\begin{array}{ccc|c} 0 & 0 & 1 & 3 \\ 0 & 2 & 0 & 2 \\ 3 & 0 & 0 & 1 \end{array}\right)$

Answers. (a) $\left(\begin{array}{ccccc|c} 1 & 0 & 3 & 4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$ (b) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right)$

3. In $\mathbb{R}^4$, let $u = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$, $v = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}$. Compute the following quantities:

(a) $||u||$

(b) $||v||$

(c) $u \cdot v$

(d) the angle between u and v

(e) $||u+v||^2$

Solutions.

(a) $||u|| = \sqrt{1^2 + (-1)^2 + 1^2 + (-1)^2} = \sqrt{4} = 2$

(b) $||v|| = \sqrt{2^2 + 2^2 + 0^2 + 0^2} = \sqrt{8} = 2\sqrt{2}$

(c) $u \cdot v = 1 \cdot 2 + (-1) \cdot 2 + 1 \cdot 0 + (-1) \cdot 0 = 0$

(d) $\cos\theta = \frac{u \cdot v}{||u|| ||v||} = \frac{0}{2 \cdot 2\sqrt{2}} = 0$, so $\theta = \pi/2$

(e) $||u+v||^2 = (1+2)^2 + (-1+2)^2 + (1+0)^2 + (-1+0)^2 = 12$

4. Solve the linear system described by the augmented matrix

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & k & 9 \end{array}\right)$$

depending on a parameter k, as a function of k. That is, specify the values of k for which a solution exists, and write a formula for the solution(s).

Solution. Performing an elementary row operation, we bring the system to an equivalent form

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & k & 9 \end{array}\right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & k-6 & -1 \end{array}\right)$$

The second row of this augmented matrix represents the equation

$$(k-6)y = -1$$

(if we think of the augmented matrix as representing a system of two equations in variables x, y). If $k = 6$, this is the equation $0 = -1$, which has no solutions, so the original system also has no solution. If $k \neq 6$, then $k-6 \neq 0$ so we can perform further elementary row operations as follows

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & k-6 & -1 \end{array}\right) \xrightarrow{\frac{1}{k-6}R_2} \left(\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 1 & -\frac{1}{k-6} \end{array}\right) \xrightarrow{R_1-3R_2} \left(\begin{array}{cc|c} 1 & 0 & 5+\frac{3}{k-6} \\ 0 & 1 & -\frac{1}{k-6} \end{array}\right)$$

which means that the system has the solutions

$$\boxed{x = 5 + \tfrac{3}{k-6}, \quad y = -\tfrac{1}{k-6} \qquad (k \neq 6).}$$

5. (a) If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation that satisfies

$$T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix},$$

find the values

$$T\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = ? \quad \text{and} \quad T\begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} = ?$$

Solution. By the linearity property of T,

$$T\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = T\left(3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) = 3T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -6 \end{pmatrix}$$

$$\begin{aligned} T\begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} &= T\left(3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = 3T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 3 \\ -6 \end{pmatrix} + \begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} \end{aligned}$$

(b) Let a transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2y+z \\ 0 \\ x+y \end{pmatrix}.$$

Find a 3×3 matrix M such that

$$T(u) = Mu$$

for any vector $u \in \mathbb{R}^3$.

Answer. $$M = \begin{pmatrix} 0 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

6. For each of the following pairs of matrices M, N, compute the matrix products MN, or specify if the product does not make sense given the dimensions of M and N.

   (a) $M = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $N = \begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \end{pmatrix}$

   (b) $M = \begin{pmatrix} 1 & 1 & x \\ -1 & -1 & -1 \end{pmatrix}$, $N = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

   (c) $M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, $N = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

   (d) $M = \begin{pmatrix} x \\ y \end{pmatrix}$, $N = \begin{pmatrix} 20 & 10 \end{pmatrix}$

   (e) $M = \begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \end{pmatrix}$, $N = \begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \end{pmatrix}$

Answers.

(a) $MN = \begin{pmatrix} 1 & 3 & 5 \\ 14 & 18 & 22 \end{pmatrix}$

(b) $MN = \begin{pmatrix} a + b + xc \\ -a - b - c \end{pmatrix}$

(c) $MN = \begin{pmatrix} 3 & 4 & 6 \\ 6 & 10 & 18 \\ 9 & 16 & 30 \end{pmatrix}$

(d) $MN = \begin{pmatrix} 20x & 10x \\ 20y & 10y \end{pmatrix}$

(e) The product is undefined because the number of columns of M is different from the number of rows of N.