Practice questions for the 1/26 midterm

Notes: The questions below represent a rough approximation to the style and difficulty level of the midterm exam. The questions should be solvable in 45 minutes (the duration of the midterm). You should not use written notes or a calculator.

1. (25 points) Find the general form of the solution of the linear system

$$\begin{cases} x_3 - 2x_4 = -2 \\ x_1 + 4x_2 + x_3 + 7x_4 = 1 \\ 2x_1 + 8x_2 + 3x_3 + 12x_4 = 0 \\ x_1 + 4x_2 - x_3 + 11x_4 = 5 \end{cases}$$

2. (15 points) For each of the following matrices, find an augmented matrix in Reduced Row Echelon Form (RREF) that is row-equivalent to it:

(a) $$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 0 \\ 0 & 1 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 \end{array}\right)$$

(b) $$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 3 \\ 0 & 2 & 0 & 2 \\ 3 & 0 & 0 & 1 \end{array}\right)$$

3. (15 points) In $\mathbb{R}^4$, let $u = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$, $v = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}$. Compute the following quantities:

   (a) $||u||$
   (b) $||v||$
   (c) $u \cdot v$
   (d) the angle between u and v
   (e) $||u+v||^2$

4. (15 points) Solve the linear system described by the augmented matrix

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & k & 9 \end{array}\right)$$

depending on a parameter k, as a function of k. That is, specify the values of k for which a solution exists, and write a formula for the solution(s).

5. (a) (8 points) If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation that satisfies

$$T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix},$$

find the values

$$T\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = ? \quad \text{and} \quad T\begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} = ?$$

(b) (7 points) Let a transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2y + z \\ 0 \\ x + y \end{pmatrix}.$$

Find a 3×3 matrix M such that

$$T(u) = Mu$$

for any vector $u \in \mathbb{R}^3$.

6. (15 points) For each of the following pairs of matrices M, N, compute the matrix products MN, or specify if the product does not make sense given the dimensions of M and N.

(a) $M = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $N = \begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \end{pmatrix}$

(b) $M = \begin{pmatrix} 1 & 1 & x \\ -1 & -1 & -1 \end{pmatrix}$, $N = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

(c) $M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, $N = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

(d) $M = \begin{pmatrix} x \\ y \end{pmatrix}$, $N = \begin{pmatrix} 20 & 10 \end{pmatrix}$

(e) $M = \begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \end{pmatrix}$, $N = \begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \end{pmatrix}$