Solutions to practice question set 2

1. (a) Define $A = \begin{pmatrix} 1 & 3 & -1 \\ 1 & 4 & -1 \\ -1 & -3 & 2 \end{pmatrix}$. Compute the inverse matrix of A.

Solution. We follow the standard method of computing the inverse matrix by performing a sequence of elementary row operations transforming the matrix into the identity matrix, and at the same time performing the same sequence of operations on the identity matrix:

$$\left(\begin{array}{ccc|ccc} 1 & 3 & -1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ -1 & -3 & 2 & 0 & 0 & 1 \end{array}\right) \overset{R_2-R_1, R_3+R_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 3 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$$
$$\overset{R_1-3R_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 4 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right) \overset{R_1+R_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -3 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$$

Thus we get the answer $A^{-1} = \begin{pmatrix} 5 & -3 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

Remarks

1. An alternative method for computing A^{-1} is by using the formula $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$.
2. In questions like this it is highly advisable to verify your answer by checking that $AA^{-1} = I$.

(b) Find the solution $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to the equation $Av = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$.

Solution.

$$v = A^{-1} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 13 \\ -3 \\ 2 \end{pmatrix}$$

2. Compute the following determinants:

(a) $\det \begin{pmatrix} 1 & 2 \\ 3 & k \end{pmatrix}$ (c) $\det \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 4 & -2 \\ 0 & 0 & 3 & 1 & 0 \end{pmatrix}$

(b) $\det \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & -2 \\ 3 & 1 & 0 \end{pmatrix}$ (d) $\det(A^3)$ where $A = \begin{pmatrix} 2 & 0 & 0 \\ 13 & 2 & 0 \\ -19 & 1001 & -1 \end{pmatrix}$

Solution.

(a) $\det \begin{pmatrix} 1 & 2 \\ 3 & k \end{pmatrix} = k - 6$

(b) $\det \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & -2 \\ 3 & 1 & 0 \end{pmatrix} = 1(4 \cdot 0 - (-2) \cdot 1) + 2(0 \cdot 1 - 3 \cdot 4) = -22$

(c) $$\det \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 4 & -2 \\ 0 & 0 & 3 & 1 & 0 \end{pmatrix} = (-1) \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 4 & -2 \\ 0 & 3 & 1 & 0 \end{pmatrix}$$

$$= (-1) \cdot 1 \cdot \det \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & -2 \\ 3 & 1 & 0 \end{pmatrix} = 22$$

(do a first row expansion twice and then use the answer to part (b))

(d) $\det(A^3) = \det(A \cdot A \cdot A) = \det(A) \cdot \det(A) \cdot \det(A) = \det(A)^3 = -64.$

(Note that $\det(A) = 2 \cdot 2 \cdot (-1) = -4$ since A is lower-triangular.)

3. Let σ be the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 6 & 5 & 4 \end{pmatrix}$ of order 6.

(a) Find the sign $\operatorname{sgn}(\sigma)$ of the permutation. Explain your answer — a guess with no explanation is not a valid answer.

Solution. σ can be obtained by starting with the identity permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$ and then doing 3 swap operations (swap 4 and 6; swap 1 and 2; swap 1 and 3). Three is an odd number so σ is an odd permutation and $\operatorname{sgn}(\sigma) = -1$.

(b) Find the inverse permutation σ^{-1}.

Solution. $\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 6 & 5 & 4 \end{pmatrix}$

4. Let M be a 4×4 matrix. Let N be the matrix obtained from M by performing the following sequence of elementary row operations:

1. Swap rows 2, 3.
2. Multiply row 2 by 2.
3. Add row 2 to row 1.

(a) If $\det(M) = 5$, find $\det(N)$.

Solution. We have $N = E_3E_2E_1M$ where E_1, E_2, E_3 are three elementary matrices corresponding to the elementary row operations described in the question. It follows that

$$\det(N) = \det(E_1)\det(E_2)\det(E_3)\det(M) = (-1)\cdot 2\cdot 1\cdot 5 = -10.$$

(b) Find a 4×4 matrix A such that $N = AM$.

Solution. By the solution to part (a) above we have

$$A = E_3E_2E_1 = E_3E_2E_1I,$$

i.e. A is the matrix obtained by performing the same sequence of row operations on I instead of M. This leads after a short computation to the answer

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

5. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by $L(v) = Av$, where $A = \begin{pmatrix} -10 & -6 \\ 18 & 11 \end{pmatrix}$. Find the two eigenvalues λ_1, λ_2 of A and for each eigenvector find an associated eigenvector.

Solution. First, we find the eigenvalues. They are the solutions to the equation $\det(\lambda I - A) = 0$. We compute:

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda + 10 & 6 \\ -18 & \lambda - 11 \end{pmatrix} = (\lambda+10)(\lambda-11) - 6(-18) = \lambda^2 - \lambda - 2$$

so we get the quadratic equation $\lambda^2 - \lambda - 2 = 0$, which has two solutions $\lambda_1 = -1$, $\lambda_2 = 2$.

The first eigenvector: To find an eigenvector $v_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ corresponding to the eigenvalue $\lambda_1 = -1$ we solve the equation

$$(A - \lambda_1 I) \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -9 & -6 \\ 18 & 12 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This has (for example) the solution $v_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, found in the usual way.

The second eigenvector: Using similar reasoning, the second eigenvector $v_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ is found by solving the equation

$$(A - \lambda_2 I) \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} -12 & -6 \\ 18 & 9 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which has the solution $v_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.