Homework Set No. 3 – Probability Theory (235A), Fall 2011

Due: Tuesday 10/18/11 at discussion section

1. Let X be an exponential r.v. with parameter λ, i.e., $F_X(x) = (1-e^{-\lambda x})1_{[0,\infty)}(x)$. Define random variables

$$\begin{aligned} Y &= \lfloor X \rfloor := \sup\{n \in \mathbb{Z} : n \le x\} && (\text{"the integer part of } X\text{"}), \\ Z &= \{X\} := X - \lfloor X \rfloor && (\text{"the fractional part of } X\text{"}). \end{aligned}$$

(a) Compute the (1-dimensional) distributions of Y and Z (in the case of Y, since it's a discrete random variable it is most convenient to describe the distribution by giving the individual probabilities $\mathbf{P}(Y = n), n = 0, 1, 2, \ldots$; for Z one should compute either the distribution function or density function).

(b) Show that Y and Z are independent. (Hint: Check that $\mathbf{P}(Y = n, Z \le t) = \mathbf{P}(Y = n)\mathbf{P}(Z \le t)$ for all n and t.)

2. (a) Let X, Y be independent r.v.'s. Define $U = \min(X, Y)$, $V = \max(X, Y)$. Find expressions for the distribution functions F_U and F_V in terms of the distribution functions of X and Y.

(b) Assume that $X \sim \mathrm{Exp}(\lambda), Y \sim \mathrm{Exp}(\mu)$ (and are independent as before). Prove that $\min(X, Y)$ has distribution $\mathrm{Exp}(\lambda + \mu)$. Try to give an intuitive explanation in terms of the kind of real-life phenomena that the exponential distribution is intended to model (e.g., measuring the time for a light-bulb to burn out, or for a radioactive particle to be emitted from a chunk of radioactive material).

(c) Let $X_1, X_2, \ldots$ be a sequence of independent r.v.'s, all of them having distribution $\mathrm{Exp}(1)$. For each $n \ge 1$ denote

$$M_n = \max(X_1, X_2, \ldots, X_n) - \log n.$$

Compute for each n the distribution function of M_n, and find the limit (if it exists)

$$F(x) = \lim_{n\to\infty} F_{M_n}(x).$$

3. If X, Y are r.v.'s with a joint density $f_{X,Y}$, the identity

$$\mathbf{P}((X,Y) \in A) = \iint_A f_{X,Y}(x,y)\,dx\,dy$$

holds for all "reasonable" sets $A \subset \mathbb{R}^2$ (in fact, for all Borel-measurable sets, but that requires knowing what that integral means for a set such as $\mathbb{R}^2 \setminus Q^2$...). In particular, if X, Y are independent and have respective densities f_X and f_Y, so $f_{X,Y}(x,y) = f_X(x) f_Y(y)$, then

$$F_{X+Y}(t) = \mathbf{P}(X + Y \le t) = \int_{-\infty}^{\infty} \int_{-\infty}^{t-x} f_X(x) f_Y(y)\,dy\,dx.$$

Differentiating with respect to t gives (assuming without justification that it is allowed to differentiate under the integral):

$$f_{X+Y}(t) = \int_{-\infty}^{\infty} f_X(x) f_Y(t-x)\,dx.$$

Use this formula to compute the distribution of $X+Y$ when X and Y are independent r.v.'s with the following (pairs of) distributions:

1. $X \sim U[0,1],\ Y \sim U[0,2]$.
2. $X \sim \mathrm{Exp}(1),\ Y \sim \mathrm{Exp}(1)$.
3. $X \sim \mathrm{Exp}(1),\ -Y \sim \mathrm{Exp}(1)$.

4. (a) Let $(A_n)_{n=1}^{\infty}$ be a sequence of events in a probability space. Show that

$$1_{\limsup A_n} = \limsup_n 1_{A_n}.$$

(The lim-sup on the left refers to the lim-sup operation on events; on the right it refers to the lim-sup of a sequence of functions; the identity is an identity of real-valued functions on Ω, i.e., should be satisfied for each individual point $\omega \in \Omega$ in the sample space). Similarly, show (either separately or by relying on the first claim) that

$$1_{\liminf A_n} = \liminf_n 1_{A_n}.$$

(b) Let U be a uniform random variable in $(0,1)$. For each $n \geq 1$ define an event A_n by

$$A_n = \{U < 1/n\}.$$

Note that $\sum_{n=1}^{\infty} \mathbf{P}(A_n) = \infty$. However, compute $\mathbf{P}(A_n \text{ i.o.})$ and show that the conclusion of the second Borel-Cantelli lemma does not hold (of course, one of the assumptions of the lemma also doesn't hold, so there's no contradiction).

5. If P, Q are two probability measures on a measurable space $(\Omega, \mathcal{F})$, we say that P **is absolutely continuous with respect to** Q, and denote this $P << Q$, if for any $A \in \mathcal{F}$, if $Q(A) = 0$ then $P(A) = 0$.

Prove that $P << Q$ if and only if for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $A \in \mathcal{F}$ and $Q(A) < \delta$ then $P(A) < \epsilon$.

Hint. Apply a certain famous lemma.

Note. The intuitive meaning of the relation $P << Q$ is as follows: suppose there is a probabilistic experiment, and we are told that one of the measures P or Q governs the statistical behavior of the outcome, but we don't know which one. (This is a situation that arises frequently in real-life applications of probability and statistics.) All we can do is perform the experiment, observe the result, and make a guess. If $P << Q$, any event which is observable with positive probability according to P also has positive Q-probability, so we can never rule out Q as the correct measure, although we may get an event with $Q(A) > 0$ and $P(A) = 0$ that enables us to rule out P. If we also have the symmetric relation $Q << P$, then we can't rule out either of the measures.