Homework Set No. 4 – Probability Theory (235A), Fall 2011

Due: 10/25/11 at discussion section

1. A function $\varphi : (a,b) \to \mathbb{R}$ is called **convex** if for any $x, y \in (a,b)$ and $\alpha \in [0,1]$ we have

$$\varphi(\alpha x + (1-\alpha)y) \le \alpha\varphi(x) + (1-\alpha)\varphi(y).$$

(a) Prove that an equivalent condition for φ to be convex is that for any $x < z < y$ in (a,b) we have

$$\frac{\varphi(z)-\varphi(x)}{z-x} \le \frac{\varphi(y)-\varphi(z)}{y-z}.$$

Deduce using the mean value theorem that if φ is twice continuously differentiable and satisfies $\varphi'' \ge 0$ then it is convex.

(b) Prove **Jensen's inequality**, which says that if X is a random variable such that $\mathbf{P}(X \in (a,b)) = 1$ and $\varphi : (a,b) \to \mathbb{R}$ is convex, then

$$\varphi(\mathbf{E}X) \le \mathbf{E}(\varphi(X)).$$

Hint. Start by proving the following property of a convex function: If φ is convex then at any point $x_0 \in (a,b)$, φ has a **supporting line**, that is, a linear function $y(x) = ax + b$ such that $y(x_0) = \varphi(x_0)$ and such that $\varphi(x) \ge y(x)$ for all $x \in (a,b)$ (to prove its existence, use the characterization of convexity from part (a) to show that the left-sided derivative of φ at x_0 is less than or equal to the right-sided derivative at x_0; the supporting line is a line passing through the point $(x_0, \varphi(x_0))$ whose slope lies between these two numbers). Now take the supporting line function at $x_0 = \mathbf{E}X$ and see what happens.

2. If X is a random variable satisfying $a \le X \le b$, prove that

$$\mathbf{V}(X) \le \frac{(b-a)^2}{4},$$

and identify when equality holds.

3. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. (independent and identically distributed) random variables with distribution $U(0,1)$. Define events $A_1, A_2, \ldots$ by

$$A_n = \{X_n = \max(X_1, X_2, \ldots, X_n)\}$$

(if A_n occurred, we say that n is a **record time**).

(a) Prove that $A_1, A_2, \ldots$ are independent events. Hint: For each $n \geq 1$, let π_n be the random permutation of $(1, 2, \ldots, n)$ obtained by forgetting the values of $(X_1, \ldots, X_n)$ and only retaining their respective order. In other words, define

$$\pi_n(k) = \#\{1 \leq j \leq n : X_j \leq X_k\}.$$

By considering the joint density $f_{X_1,\ldots,X_n}$ (a uniform density on the n-dimensional unit cube), show that π_n is a uniformly random permutation of n elements, i.e. $\mathbf{P}(\pi_n = \sigma) = 1/n!$ for any permutation $\sigma \in S_n$. Deduce that the event $A_n = \{\pi_n(n) = n\}$ is independent of π_{n-1} and therefore is independent of the previous events $(A_1, \ldots, A_{n-1})$, which are all determined by π_{n-1}.

(b) Define

$$R_n = \sum_{k=1}^{n} 1_{A_k} = \#\{1 \leq k \leq n : k \text{ is a record time}\}, \qquad (n = 1, 2, \ldots).$$

Compute $\mathbf{E}(R_n)$ and $\mathbf{V}(R_n)$. Deduce that if $(m_n)_{n=1}^{\infty}$ is a sequence of positive numbers such that $m_n \uparrow \infty$, however slowly, then the number R_n of record times up to time n satisfies

$$\mathbf{P}\left(|R_n - \log n| > m_n\sqrt{\log n}\right) \xrightarrow[n\to\infty]{} 0.$$

4. Compute $\mathbf{E}(X)$ and $\mathbf{V}(X)$ when X is a random variable having each of the following distributions:

1. $X \sim \text{Binomial}(n, p)$.
2. $X \sim \text{Poisson}(\lambda)$, i.e., $\mathbf{P}(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}$, $(k = 0, 1, 2, \ldots)$.
3. $X \sim \text{Geom}(p)$, i.e,. $\mathbf{P}(X = k) = p(1-p)^{k-1}$, $(k = 1, 2, \ldots)$.
4. $X \sim U\{1, 2, \ldots, n\}$ (the discrete uniform distribution on $\{1, 2, \ldots, n\}$).
5. $X \sim U(a, b)$ (the uniform distribution on the interval (a, b)).
6. $X \sim \text{Exp}(\lambda)$

5. (a) If X, Y are independent r.v.'s taking values in $\mathbb{Z}$, show that

$$\mathbf{P}(X+Y=n) = \sum_{k=-\infty}^{\infty} \mathbf{P}(X=k)\mathbf{P}(Y=n-k) \qquad (n \in \mathbb{Z})$$

(compare this formula with the convolution formula in the case of r.v.'s with density).

(b) Use this to show that if $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ are independent then $X+Y \sim \text{Poisson}(\lambda+\mu)$. (Recall that for a parameter $\lambda > 0$, we say that $X \sim \text{Poisson}(\lambda)$ if $\mathbf{P}(X=k) = e^{-\lambda}\lambda^k/k!$ for $k = 0, 1, 2, \ldots$).

(c) Use the same "discrete convolution" formula to prove directly that if $X \sim \text{Bin}(n,p)$ and $Y \sim \text{Bin}(m,p)$ are independent then $X+Y \sim \text{Bin}(n+m,p)$. You may make use of the combinatorial identity (known as the Vandermonde identity or Chu-Vandermonde identity)

$$\sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} = \binom{n+m}{k}, \qquad (n, m \geq 0,\ 0 \leq k \leq n+m).$$

As a bonus, try to find a direct combinatorial proof for this identity. An amusing version of the answer can be found at:

`http://en.wikipedia.org/wiki/Vandermonde's_identity`.

6. Prove that if X is a random variable that is independent of itself, then X is a.s. constant, i.e., there is a constant $c \in \mathbb{R}$ such that $\mathbf{P}(X=c) = 1$.