Homework Set No. 5 – Probability Theory (235A), Fall 2011

Due: Tuesday 11/01/11 at discussion section

1. (a) If $X \geq 0$ is a nonnegative r.v. with distribution function F, show that

$$\mathbf{E}(X) = \int_0^\infty \mathbf{P}(X \geq x)\, dx.$$

(b) Prove that if $X_1, X_2, \ldots,$ is a sequence of independent and identically distributed ("i.i.d.") r.v.'s, then

$$\mathbf{P}(|X_n| \geq n \text{ i.o.}) = \begin{cases} 0 & \text{if } \mathbf{E}|X_1| < \infty, \\ 1 & \text{if } \mathbf{E}|X_1| = \infty. \end{cases}$$

(c) Deduce the following converse to the Strong Law of Large Numbers in the case of undefined expectations: If $X_1, X_2, \ldots$ are i.i.d. and $\mathbf{E}X_1$ is undefined (meaning that $\mathbf{E}X_{1+} = \mathbf{E}X_{1-} = \infty$) then

$$\mathbf{P}\left(\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n X_k \text{ does not exist}\right) = 1.$$

2. Let X be a r.v. with finite variance, and define a function $M(t) = \mathbf{E}|X - t|$, the "mean absolute deviation of X from t". The goal of this question is to show that the function $M(t)$, like its easier to understand and better-behaved cousin, $\mathbf{E}(X - t)^2$ (the "moment of inertia" around t, which by the Huygens-Steiner theorem is simply a parabola in t, taking its minimum value of $\mathbf{V}(X)$ at $t = \mathbf{E}X$), also has some unexpectedly nice propreties.

(a) Prove that $M(t) \geq |t - \mathbf{E}X|$.

(b) Prove that $M(t)$ is a convex function.

(c) Prove that

$$\int_{-\infty}^\infty \Big(M(t) - |t - \mathbf{E}X|\Big)dt = \mathbf{V}(X)$$

(see hints below). Deduce in particular that $M(t) - |t - \mathbf{E}X| \xrightarrow[t\to\pm\infty]{} 0$ (again under the assumption that $\mathbf{V}(X) < \infty$). If it helps, you may assume that X has a density f_X.

(d) Prove that if t_0 is a (not necessarily unique) minimum point of $M(t)$, then t_0 is a median (that is, a 0.5-quantile) of X.

(e) Optionally, draw (or, at least, imagine) a diagram showing the graphs of the two functions $M(t)$ and $|t - \mathbf{E}X|$ illustrating schematically the facts (a)–(d) above.

Hints: For (c), assume first (without loss of generality - why?) that $\mathbf{E}X = 0$. Divide the integral into two integrals, on the positive real axis and the negative real axis. For each of the two integrals, by decomposing $|X - t|$ into a sum of its positive and negative parts and using the fact that $\mathbf{E}X = 0$ in a clever way, show that one may replace the integrand $(\mathbf{E}|X - t| - |t|)$ by a constant multiple of either $\mathbf{E}(X - t)_+$ or $\mathbf{E}(X - t)_-$, and proceed from there.

For (d), first, develop your intuition by plotting the function $M(t)$ in a couple of cases, for example when $X \sim \text{Binom}(1, 1/2)$ and when $X \sim \text{Binom}(2, 1/2)$. Second, if $t_0 < t_1$, plot the graph of the function $x \to \frac{|x-t_1|-|x-t_0|}{t_1-t_0}$, and deduce from this a formula for $M'(t_0+)$ and (by considering $t_1 < t_0$ instead) a similar formula for $M'(t_0-)$, the right- and left-sided derivatives of M at t_0, respectively. On the other hand, think how the condition that t_0 is a minimum point of $M(t)$ can be expressed in terms of these one-sided derivatives.

3. (a) Let $\Gamma(t)$ denote the *Euler gamma function*, defined by

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\, dx, \qquad (t > 0).$$

Show that the special value $\Gamma(1/2) = \sqrt{\pi}$ of the gamma function is equivalent to the integral evaluation $\sqrt{2\pi} = \int_{-\infty}^{\infty} e^{-x^2/2}\, dx$ (which is equivalent to the standard normal density being a density function).

(b) Prove that the Euler gamma function satisfies for all $t > 0$ the identity

$$\Gamma(t + 1) = t\,\Gamma(t).$$

(This identity immediately implies the fact that $\Gamma(n + 1) = n!$ for integer $n \geq 0$.)

(c) Find a formula for the values of $\Gamma(\cdot)$ at half-integers, that is,

$$\Gamma\left(n + \tfrac{1}{2}\right) = ?, \qquad (n \geq 0).$$

4. Compute $\mathbf{E}X^n$ when $n \geq 0$ is an integer and X has each of the following distributions:

(a) $X \sim U(a, b)$

(b) $X \sim \text{Exp}(\lambda)$

(c) $X \sim \text{Gamma}(\alpha, \lambda)$, i.e. $f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1}, \quad (x > 0)$.

(d) $X \sim \text{Beta}(a, b)$, i.e. $f_X(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}, \quad (0 < x < 1)$, where

$$B(a, b) = \int_0^1 u^{a-1}(1-u)^{b-1}\, du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is the *Euler beta function*.

(e) $X \sim N(0, 1)$. In this case, identify $\mathbf{E}X^n$ combinatorially as the number of **matchings** of a set of size n into pairs (for example, if a university dorm has only 2-person housing units, then when n is even this is the number of ways to divide n students into pairs of roommates; no importance is given to the ordering of the pairs).

(f) (Optional, and more difficult) $X \sim N(1, 1)$. In this case, identify $\mathbf{E}X^n$ combinatorially as the number of **involutions** (permutations which are self-inverse) of a set of n elements. To count the involutions, it is a good idea to divide them into classes according to how many fixed points they have. (Note: the expression for $\mathbf{E}(X^n)$ may not have a very simple form.)