Homework Set No. 7 – Probability Theory (235A), Fall 2011

Due: 11/15/11

1. Prove that if F and $(F_n)_{n=1}^{\infty}$ are distribution functions, F is continuous, and $F_n(t) \to F(t)$ as $n \to \infty$ for any $t \in \mathbb{R}$, then the convergence is uniform in t.

2. Let $\varphi(x) = (2\pi)^{-1/2} e^{-x^2/2}$ be the standard normal density function.

(a) If $X_1, X_2, \ldots$ are i.i.d. Poisson(1) random variables and $S_n = \sum_{k=1}^{n} X_k$ (so $S_n \sim$ Poisson(n)), show that if n is large and k is an integer such that $k \approx n + x\sqrt{n}$ then

$$\mathbf{P}(S_n = k) \approx \frac{1}{\sqrt{n}} \varphi(x).$$

Hint: Use the fact that $\log(1+u) = u - u^2/2 + O(u^3)$ as $u \to 0$.

(b) Find $\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!}$.

(c) If $X_1, X_2, \ldots$ are i.i.d. Exp(1) random variables and denote $S_n = \sum_{k=1}^{n} X_k$ (so $S_n \sim$ Gamma$(n,1)$), $\hat{S}_n = (S_n - n)/\sqrt{n}$. Show that if n is large and $x \in \mathbb{R}$ is fixed then the density of $\hat{S}_n$ satisfies

$$f_{\hat{S}_n}(x) \approx \varphi(x).$$

3. **(a)** Prove that if $X, (X_n)_{n=1}^{\infty}$ are random variables such that $X_n \to X$ in probability then $X_n \implies X$.

(b) Prove that if $X_n \implies c$ where $c \in \mathbb{R}$ is a constant, then $X_n \to c$ in probability.

(c) Prove that if $Z, (X_n)_{n=1}^{\infty}, (Y_n)_{n=1}^{\infty}$ are random variables such that $X_n \implies Z$ and $X_n - Y_n \to 0$ in probability, then $Y_n \implies Z$.

4. **(a)** Let $X, (X_n)_{n=1}^{\infty}$ be integer-valued r.v.'s. Show that $X_n \implies X$ if and only if $\mathbf{P}(X_n = k) \to \mathbf{P}(X = k)$ for any $k \in \mathbb{Z}$.

(b) If $\lambda > 0$ is a fixed number, and for each n, Z_n is a r.v. with distribution Binomial$(n, \lambda/n)$, show that

$$Z_n \implies \text{Poisson}(\lambda).$$

5. Let $f(x) = (2\pi)^{-1/2} e^{-x^2/2}$ be the density function of the standard normal distribution, and let $\Phi(x) = \int_{-\infty}^{x} f(u)\,du$ be its c.d.f. Prove the inequalities

$$\frac{1}{x + x^{-1}} f(x) \le 1 - \Phi(x) \le \frac{1}{x} f(x), \qquad (x > 0). \tag{1}$$

Note that for large x this gives a very accurate two-sided bound for the tail of the normal distribution. In fact, it can be shown that

$$1 - \Phi(x) = f(x) \cdot \cfrac{1}{x + \cfrac{1}{x + \cfrac{2}{x + \cfrac{3}{x + \cfrac{4}{x + \ldots}}}}}$$

which gives a relatively efficient method of estimating $\Phi(x)$.

Hint: To prove the upper bound in (1), use the fact that for $t > x$ we have $e^{-t^2/2} \le (t/x)e^{-t^2/2}$. For the lower bound, use the identity

$$\frac{d}{dx}\left(\frac{e^{-x^2/2}}{x}\right) = -\left(1 + \frac{1}{x^2}\right) e^{-x^2/2}$$

to compute $\int_x^\infty (1 + u^{-2}) e^{-u^2/2}\,du$. On the other hand, show that this integral is bounded from above by $(1 + x^{-2}) \int_x^\infty e^{-u^2/2}\,du$.