MAT235: Discussion 1

9/30/13

1 Moments of Discrete Random Variables

$E(X^n) = \sum x^n f(x)$ where f is the probability function of X.

Example: Find the expectation of $X \sim Bin(n, p)$.

Method 1: Using the definition of expectation.

$$E(X) = \sum_{k=0}^{n} k f(k) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}.$$

Notice that, by differentiating the identity w.r.t. x

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n,$$

and multiply by x to obtain

$$\sum_{k=0}^{n} k \binom{n}{k} x^k = nx(1+x)^{n-1}.$$

Substitute $x = p/(1-p)$ to obtain $E(X) = np$.

Method 2: X is the sum of n independent Bernoulli p random variables each has expectation p. Hence $E(X) = np$.

Try geometric, negative binomial, Poisson random variables.

2 Some Properties of Discrete Random Variables

Example: $X \sim Poi(\lambda)$ and $Y \sim Poi(\mu)$ are independent, then $X + Y \sim Poi(\lambda + \mu)$.

$$\begin{aligned}
P(X+Y=k) &= \sum_{i=0}^{k} P(X=i, Y=k-i) = \sum_{i=0}^{k} P(X=i)P(Y=k-i) \\
&= \sum_{i=0}^{k} \frac{\lambda^i e^{-\lambda}}{i!} \frac{\mu^{k-i} e^{-\mu}}{(k-i)!} = \frac{e^{-(\lambda+\mu)}}{k!} \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} \lambda^i \mu^{k-i} \\
&= \frac{e^{-(\lambda+\mu)}}{k!} (\lambda+\mu)^k.
\end{aligned}$$

Try to show if $X \sim Bin(n, p)$ and $Y \sim Bin(m, p)$ are independent, then $X + Y \sim Bin(m + n, p)$, and $X \sim NB(n, p)$ and $Y \sim NB(m, p)$ are independent, then $X + Y \sim NB(m + n, p)$.

Example: Properties of geometric random variables.
Memoryless property: Suppose $X \sim Geo(p)$, then $P(X > s + t | X > t) = P(X > s)$.

$$P(X > s+t | X > t) = \frac{P(X > s+t)}{P(X > t)} = \frac{(1-p)^{s+t}}{(1-p)^t} = (1-p)^s.$$

Suppose $X \sim Geo(p)$ and $Y \sim Geo(q)$ are independent, then $\min(X, Y) \sim Geo(p + q - pq)$.
Let $Z = \min(X, Y)$

$P(Z > k) = P(\min(X, Y) > k) = P(\{X > k\} \cup \{Y > k\}) = (1 - p)^{k-1}(1 - q)^{k-1} = (1 - p - q + pq)^{k-1}$.

3 Independence of Discrete Random Variables

Discrete random variables X and Y are independent if the events $\{X = x\}$ and $\{Y = y\}$ are independent for all x and y.

Example: (Poisson Splitting) Suppose a coin is tossed and heads turns up w.p. p. Let X and Y be the numbers of heads and tails respectively. It is no surprise that X and Y are not independent. Suppose now the coin is tossed a random number N of times, where $N \sim Poi(\lambda)$. Then $X \sim Poi(\lambda p)$, $Y \sim Poi(\lambda(1 - p))$ and they are independent.

$$
\begin{aligned}
P(X = x) &= \sum_{n \geq x} P(X = x | N = n) P(N = n) \\
&= \sum_{n \geq x} \binom{n}{x} p^x (1 - p)^{(n-x)} \frac{e^{-\lambda} \lambda^n}{n!} = \frac{e^{-\lambda p} (\lambda p)^x}{x!}
\end{aligned}
$$

Hence $X \sim Poi(\lambda p)$, and similarly $Y \sim Poi(\lambda(1-p))$. To show independence, one needs to check that

$$P(X = x, Y = y) = P(X = x)P(Y = y).$$

This is left as a homework problem.

4 Poisson Distribution as Limits of Binomials

Suppose λ is a constant and $X \sim Bin(n, p = \lambda/n)$, then $P(X = k) \to P(Y = k)$ as $n \to \infty$, where $Y \sim Poi(\lambda)$.

$$
\begin{aligned}
P(X = k) &= \frac{n!}{k!(n - k)!} \big(\frac{\lambda}{n}\big)^k \big(\frac{n - \lambda}{n}\big)^{n-k} \\
&= \frac{\lambda^k}{k!} \Big[n(n - 1) \cdots (n - k + 1) \frac{1}{n^k} \big(\frac{n - \lambda}{n}\big)^{n-k}\Big] \\
\lim_{n \to \infty} P(X = k) &= \frac{\lambda^k}{k!} \cdot \lim_{n \to \infty} \Big[n(n - 1) \cdots (n - k + 1) \frac{1}{n^k} \big(\frac{n - \lambda}{n}\big)^{n-k}\Big] \\
&= \frac{\lambda^k}{k!} \cdot \lim_{n \to \infty} \frac{n(n - 1) \cdots (n - k + 1)}{n^k} \cdot \lim_{n \to \infty} \big(1 - \frac{\lambda}{n}\big)^{-k} \cdot \lim_{n \to \infty} \big(1 - \frac{\lambda}{n}\big)^n,
\end{aligned}
$$

where the first two limits goes to 1 and the last limit goes to e^{λ}.

5 Application of inclusion-exclusion formula

What is the probability of randomly putting n balls into k bins ($k \leq n$) such that there is no empty bin?
Let A_i denote the event that at least i bins are empty, then $P(A_i) = \binom{k}{i} \big(\frac{k-i}{k}\big)^n$.

$$P(\text{no empty bin}) = 1 - p(\cup_{i=1}^{k-1} A_i) = \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} \big(\frac{k - i}{k}\big)^n.$$