MAT235: Discussion 2

10/7/13

1 Construction of non-measurable set

For probability measure there are some properties we would like it to have. For example, we would like the probability of the union of a finite or infinite sequence of disjoints sets to be equal to the sum of the probability of the individual set. And we would like that it is unchanged under translation, rotation or reflection of the set. But these conditions are mutually inconsistent in general.
Example: Consider $[0,1)$ where we would like the probability measure is just the length of the set (interval). Define a equivalence relation by declaring that $x \sim y$ if and only if $x - y$ is rational. If we believe in axiom of choice, let N be a subset of $[0,1)$ that contains precisely one member of each equivalent class. Let Q denote the set of all rational numbers in $[0,1)$, and for each $q \in Q$ let $N_q = \{x + q : x \in N \cap [0, 1-q)\} \cup \{x + q - 1 : x \in N \cap [1-q, 1)\}$. Then we have $P(N_q) = P(N)$ for all $q \in Q$. And $\cup_{q\in Q} N_q = [0,1)$ where Q is countable and N_q are disjoint for different q. Thus we should have:

$$P([0,1)) = \sum_{q\in Q} N_q$$

. But the RHS is either 0 or ∞ which both disagree with LHS.

2 Moments of Continuous Random Variables

$E(X^n) = \int x^n f(x)$ where f is the probability density function of X.

Example: Find the expectation of $X \sim N(\mu, \sigma^2)$.

Using the definition of expectation.

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f(x) dx = \int x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \int (x-\mu) f(x) dx + \int \mu f(x) dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int (x-\mu) e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx + \mu \int f(x) dx \quad \text{let } w = x - \mu \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int \frac{1}{2} w e^{-\frac{w^2}{2\sigma^2}} dw + \mu \\
&= \mu,
\end{aligned}$$

where it is easy to see that the first term in the second last line is 0 by symmetry.
Try exponential, uniform, Beta random variables.

3 Some Properties of Continuous Random Variables

Example: $X \sim Exp(\lambda)$ and $Y \sim Exp(\mu)$ are independent, then $\min\{X, Y\} \sim Exp(\lambda + \mu)$. Let $Z = \min(X, Y)$

$$P(Z > k) = P(\min(X,Y) > k) = P(\{X > k\} \cup \{Y > k\}) = e^{-\lambda x} e^{-\mu x} = e^{-(\lambda+\mu)x}.$$

Memoryless property of exponential distribution: Suppose $X \sim Exp(\lambda)$, then $P(X > s+t|X > t) = P(X > s)$.

$$P(X > s+t|X > t) = \frac{P(X > s+t)}{P(X > t)} = \frac{e^{-\lambda(s+t)}}{e^{-t}} = e^{-\lambda s}.$$

Suppose $X_i \sim U(0,1)$ are independent, let $Z = \min\{X_1, \cdots, X_n\}$, then $\lim_{n\to\infty} nZ$ is $Exp(1)$.

$$P(nZ > x) = P(X_1 > \frac{x}{n}) \cdots P(X_n > \frac{x}{n}) = (1 - \frac{x}{n})^n, \lim_{n\to\infty} P(nZ > x) = e^{-x}.$$

4 Independence of Discrete Random Variables

Continuous random variables X and Y are independent if the events $\{X \leq x\}$ and $\{Y \leq y\}$ are independent for all x and y. If X and Y are independent, then so are $g(X)$ and $g(Y)$.

Example: X and Y are independent and U is a r.v. taking ± 1 w.p. $1/2$. Define $S = UX, T = UY$. Clearly if X and Y are positive, then S positive implies T positive. Hence S and T are not independent. However S^2 and T^2 are independent, because $S^2 = X^2$ and $T^2 = Y^2$.

Example: A plane is ruled by the lines $y = 0, \pm 1, \pm 2, \cdots$ and a needle of length 1 is cast randomly on to the plane. What is the probability tat it intersects some line? We suppose the needle shows no preference for position or direction.
Solution: Let (X, Y) be the coordinates of the center of the needle and let Θ be the angle made by the needle and the x-axis modulo π. Denote the distance from the needle's center and the nearest line beneath it by $Z = Y - \lfloor Y \rfloor$. Since the needle is casted randomly and has no preference for position or direction, it means that:

- Z is uniformly distributed on $[0,1]$, so that $f_Z(z) = 1$ if $0 \leq z \leq 1$,
- Θ is uniformly distributed on $[0, \pi]$, so that $f_\Theta(\theta) = 1/\pi$ if $0 \leq \theta \leq \pi$,
- Z and Θ are independent, so that $f_{Z,\Theta}(z, \theta) = f_Z(z) f_\Theta(\theta)$.

Thus the pair Z, Θ has joint density function $f(z, \theta) = 1/\pi$ for $0 \leq z \leq 1$, $0 \leq \theta \leq \pi$. Draw a diagram to see that an intersection occurs if and only if $(Z, \Theta) \in B$ where $B = \{(z, \theta) : z \leq \frac{1}{2}\sin\theta \text{ or } 1 - z \leq \frac{1}{2}\sin\theta\}$. Hence

$$P(\text{intersection}) = \int\int_B f(z,\theta) dz d\theta = \frac{1}{\pi}\int_0^\pi \Big(\int_0^{\frac{1}{2}\sin\theta} dz + \int_{1-\frac{1}{2}\sin\theta}^1 dz\Big) d\theta = \frac{2}{\pi}.$$

We can use this result to manually estimate the value of π.

5 Generating random variables

Example: Suppose we can generate $U \sim U(0,1)$, how to generate r.v. X which has a 1-to-1 c.d.f. F. Since F is 1-to-1, it has an inverse F^{-1}. For r.v. $F^{-1}(U)$, we have:

$$P(F^{-1}(U) \leq x) = P(U \leq F(x)) = F(x)$$

. Therefore $F^{-1}(U)$ is the desired random variable.

Example: Generating Bernoulli(p) r.v. using $U(0,1)$.
Set $X = 1$ if $u \leq p$ and $X = 0$ o.w..

Example : How to simulate an unbiased coin using a biased coin that heads come up w.p. p.
Notice that toss a biased coin twice, in the sample space {HH,HT,TH,TT}, HT and TH have the same probability.

Example: How to simulate an biased coin that heads come up w.p. p using an unbiased coin.
Let $0.p_1 p_2 \cdots$ be the base-2 representation of p. We toss the unbiased coin infinitely many times, let $X = 1$ if head comes up and $X = 0$ if tail comes up. If $\sum_{k=1}^{\infty} \frac{X_k}{2^k} \leq p$ we consider that we have a head for the biased coin. This is equivalent as if $X_k > p_k$ for $k = 1, 2, \cdots$, then we have a tail for the biased coin.