MAT235: Discussion 3

1 Convolution

Joint density:

$$P((X,Y)\in A)=\int\int_A f_{X,Y}(x,y)dxdy.$$

Convolution of X, Y is

$$f_{X+Y}(z)=\int f(x,z-x)dx.$$

If X and Y are independent, then

$$f_{X+Y}(z)=\int f_X(x)f_Y(z-x)dx=\int f_X(z-y)f_Y(y)dy.$$

Example: Find the density function of $Z = X + Y$ when X and Y have joint density function $f(x,y)=\frac{1}{2}(x+y)e^{-(x+y)}, x,y\geq 0$.

$$f_Z(z)=\int_0^z \frac{1}{2}ze^{-z}dx=\frac{1}{2}z^2e^{-z}.$$

Example: Suppose X and Y are independent $U(0,1)$, then $Z = X + Y$ has density

$$\begin{aligned} f_Z(z)&=\int I_{x\in(0,1)}I_{z-x\in(0,1)}dx=\begin{cases}\int_0^z dx & \text{if } z\in(0,1)\\ \int_{z-1}^1 dx & \text{if } z\in(1,2)\end{cases}\\ &=\begin{cases} z & \text{if } z\in(0,1)\\ 2-z & \text{if } z\in(1,2).\end{cases}\end{aligned}$$

Example: Suppose $X\sim Gamma(\alpha,\lambda)$, $Y\sim Gamma(\beta,\lambda)$ are independent, then $Z=X+Y$ is $Gamma(\alpha+\beta,\lambda)$.

$$\begin{aligned} f_Z(z)&=\int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}\frac{\lambda^\beta}{\Gamma(\beta)}x^{\beta-1}e^{-\lambda x}dx=\frac{\lambda^{\alpha+\beta}e^{-\lambda z}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^z x^{\alpha-1}(z-x)^{\beta-1}dx\\ &=\frac{\lambda^{\alpha+\beta}e^{-\lambda z}}{\Gamma(\alpha)\Gamma(\beta)}z^{\alpha+\beta-1}\int_0^1(\frac{x}{z})^{\alpha-1}(1-\frac{x}{z})^{\beta-1}d(x/z)\\ &=\frac{\lambda^{\alpha+\beta}e^{-\lambda z}B(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)}z^{\alpha+\beta-1}\int_0^1\frac{(w)^{\alpha-1}(1-w)^{\beta-1}}{B(\alpha,\beta)}dw\\ &=\frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha+\beta)}e^{-\lambda z}z^{\alpha+\beta-1}.\end{aligned}$$

Here we used the density of Beta distribution and $B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

Example: Suppose $X_1,\cdots,X_n\sim_{i.i.d.} Exp(\lambda)$, then $\max(X_1,\cdots,X_n)=^d X_1+\frac{X_2}{2}+\cdots+\frac{X_n}{n}$. This can be shown using induction and density of maximum statistics. For $n=2$,

$$\begin{aligned} P(\max(X_1,X_2)<z)&=P(X_1<z)P(X_2<z)=(1-e^{-\lambda z})^2=1-2e^{-\lambda z}+e^{-2\lambda z}.\\ f_{\max(X_1,X_2)}(z)&=2\lambda e^{-\lambda z}-2\lambda e^{-2\lambda z}.\\ f_{X_1+X_2/2}(z)&=\int_0^z \lambda e^{-\lambda x}2\lambda e^{-2\lambda(z-x)}dx=2\lambda e^{-\lambda z}-2\lambda e^{-2\lambda z}.\end{aligned}$$

2 Transformation of random vector

If X_1, X_2 have joint density function f, then the pair Y_1, Y_2 given by $(Y_1, Y_2) = T(X_1, X_2)$ has joint density function

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} f(x_1(y_1,y_2), x_2(y_1,y_2))|J(y_1,y_2)| & \text{if } (y_1,y_2) \text{ is in the range of } T, \\ 0 \quad \text{otherwise,} \end{cases}$$

where J is the Jacobian matrix of $T^{-1}(y_1, y_2)$.

Example (Polar decomposition): Suppose X, Y are independent $N(0,1)$. We can write $X = R\cos\Theta, Y = R\sin\Theta$ for some r.v. R and Θ. Then we can compute the joint density of (R, Θ).

$$|J| = det\begin{pmatrix} \cos\Theta & \sin\Theta \\ -R\sin\Theta & R\cos\Theta \end{pmatrix} = R$$

$$\begin{aligned} f_{R,\Theta}(r,\theta) &= f_X(r\cos\theta) f_Y(r\sin\theta) r = re^{-r^2/2}\frac{1}{2\pi} \\ &= re^{-r^2/2} \cdot \frac{1}{2\pi} = f_R(r) \cdot f_\Theta(\theta). \end{aligned}$$

Here we used the fact that if X, Y are continuous random variables, then X, Y are independent if and only if $f_{X,Y}(x,y)$ can be factorized into $g(x)h(y)$. You can try to use this claim. The result of the above computation suggests an algorithm to generate independent Normal random variables.

Example: Let X and Y be independent exponential random variables with parameter 1. Find the joint density function of $U = X + Y$ and $V = X/(X+Y)$, and deduce that V is uniformly distributed on [0,1].
The transformation $X = UV, Y = U - UV$ has Jacobian

$$|J| = det\begin{pmatrix} V & U \\ 1-V & -U \end{pmatrix}$$

$$f_{U,V}(u,v) = e^{-uv}e^{u+uv}u = ue^{-u}.$$

By similar arguments as in last example, we can show that V is uniformly distributed on [0,1].

Example: Suppose $X \sim Gamma(\alpha, \lambda), Y \sim Gamma(\beta, \lambda)$ are independent. Derive the joint distribution of $U = \frac{X}{X+Y}, V = X + Y$ and show that $U \sim Beta(\alpha, \beta), V \sim Gamma(\alpha + \beta, \lambda)$ and U, V are independent.

$$X = UV, Y = V - UV \Rightarrow J = \begin{pmatrix} V & U \\ -V & 1-U \end{pmatrix} \Rightarrow |J| = V$$

$$\begin{aligned} f_{U,V}(u,v) &= \frac{\lambda^\alpha}{\Gamma(\alpha)}(uv)^{\alpha-1}e^{-\lambda uv} \cdot \frac{\lambda^\beta}{\Gamma(\beta)}(v-uv)^{\beta-1}e^{-\lambda(v-uv)} \cdot v \\ &= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha+\beta)}e^{-\lambda v}v^{\alpha+\beta-1} \cdot \frac{1}{B(\alpha,\beta)}u^{\alpha-1}(1-u)^{\beta-1}. \end{aligned}$$