MAT235: Discussion 5

1 Simple random walk

Let $\{(S_n^d)_{n=0}^\infty\}$ be the simple symmetric random walk on Z^d. That is $S_0^d = 0, S_n^d = \sum_{k=1}^n X_k$ where $X_1, X_2, \cdots$ are i.i.d. d-dimensional random vectors distributed uniformly on the $2d$ points $\{\pm e_j : j = 1, \cdots, d\}$. Denote

- $E_{return} = \{S_n^d = 0 \text{ for some } n \geq 1\}$,
- $E_{recurrence} = \{S_n^d = 0 \text{ i.o.}\}$.

Theorem: (1): $P(E_{return}) = 1$ for $d = 1, 2$; $P(E_{return}) < 1$ for $d \geq 3$.
(2): $P(E_{recurrence}) = 1$ for $d = 1, 2$; $P(E_{recurrence}) = 0$ for $d \geq 3$.

Lemma 1: The 1-dim simple random walk is spatially homogeneous. That is

$$P(S_n = j | S_0 = 0) = P(S_n = j + b | S_0 = b).$$

Proof: Both sides equal $P(\sum_1^n X_i = j)$.

Lemma 2: The 1-dim simple random walk is temporally homogeneous. That is

$$P(S_n = j | S_0 = 0) = P(S_{m+n} = j | S_m = 0).$$

Proof:

$$LHS = P(\sum_1^n X_i = j) = P(\sum_{m+1}^{m+n} X_i = j) = RHS.$$

Let $u_n = P(S_n = 0), u_0 = 1$ be the probability of being at the origin after n steps, and let $f_n = P(S_n = 0, S_1 \neq 0, \cdots, S_{n-1} \neq 0), f_0 = 0$ be the probability that the first return occurs after n steps. Denote the generating sequence of these sequences by

$$U(x) = \sum_{n=0}^\infty u_n x^n \quad F(x) = \sum_{n=0}^\infty f_n x^n \quad |x| < 1.$$

Note that $P(E_{return}) = F(1) = \lim_{x \to 1} F(x)$ by Abel's Theorem.

Lemma 3: $P(E_{return}) < 1$ iff $\sum_{n=0}^\infty u_n < \infty$.
Proof: Let A be the event that $S_n = 0$, and let B_k be the event that the first return to the origin happens at k-th step. Clearly the B_k are disjoint, so:

$$P(A) = \sum_{k=1}^n P(A|B_k) P(B_k).$$

And $P(B_k) = f_k$ and $P(A|B_k) = u_{n-k}$ by temporal homogeneity. Therefore

$$u_n = \sum_{k=1}^n u_{n-k} f_k \quad \text{for } n \geq 1.$$

Multiply the above equation by x^n, sum over n and remembering $u_0 = 1$, we can obtain:

$$U(x) - 1 = U(x)F(x) \Rightarrow U(x) = \frac{1}{1 - F(x)}.$$

From this we see that $\lim_{x\to 1} F(x) < 1$ if and only if $\lim_{x\to 1} U(x) < \infty$. Since $U(1) = \sum_{n=0}^{\infty} u_n = \lim_{x\to 1} U(x)$, this happens if and only if $\sum_{n=0}^{\infty} u_n < \infty$. Thus to see whether $P(E_{return}) < 1$, it suffices to determine the convergence of the series $\sum_n P(S_n^d = 0)$.

Proof of Theorem (1): For $d = 1$,

$$P(S_{2n}^1 = 0) = \binom{2n}{n}\frac{1}{2^n} \sim \frac{1}{\sqrt{\pi n}}$$

by Stirling's formula. Thus $\sum P(S_n^1 = 0) = \infty$, which implies $P(E_{return}) = 1$.
For $d = 2$,

$$\begin{aligned}P(S_{2n}^2 = 0) &= P(\text{for some } 0 \le m \le n,\ m \text{ steps up and } m \text{ steps down, } n-m \text{ steps left and } n-m \text{ steps right})\\ &= \sum_{m=0}^{n} \frac{2n!}{m!m!(n-m)!(n-m)!}\frac{1}{4^{2n}} = 4^{-2n}\binom{2n}{n}\sum_{m=0}^{n}\binom{n}{m}^2 = 4^{-2n}\binom{2n}{n}^2\\ &= P(S_n^1 = 0)^2 \sim \sqrt{\pi n}.\end{aligned}$$

Again $\sum P(S_n^2 = 0) < \infty$. Therefore $P(E_{return}) = 1$.
For $d = 3$,

$$\begin{aligned}P(S_{2n}^3 = 0) &= \sum_{l,m=0}^{l+m=n} \frac{2n!}{(l!m!(n-l-m)!)^2}\frac{1}{6^{2n}} = 2^{-2n}\binom{2n}{n}\sum_{l,m}\Big(\frac{n!}{l!m!(n-l-m)!}3^{-n}\Big)^2\\ &\le 2^{-2n}\binom{2n}{n}\Big[\sum_{l,m}\Big(\frac{n!}{l!m!(n-l-m)!}3^{-n}\Big)\Big](\max_{l,m} 3^{-n}\frac{n!}{l!m!(n-l-m)!}),\end{aligned}$$

where the term in the square brackets is 1. Because it is the "factorial distribution" (random coloring of n balls using 3 colors). And the maximum of the last term is obtained when $l, m, n-l-m$ are as close as possible to $1/3$. In that case we have:

$$\frac{n!}{l!m!(n-l-m)!} \approx C\frac{\sqrt{n}}{(\sqrt{n/3})^3}\frac{n^n}{\big[(\sqrt{n/3})^{n/3}\big]^3} = 3^n\frac{C}{n}.$$

Together with $2^{-2n}\binom{2n}{n} \sim C/\sqrt{n}$, we have

$$P(S_{2n}^3 = 0) \le \frac{C}{n^{3/2}},$$

which implies $\sum P(S_{2n}^3 = 0) < \infty$, therefore $P(E_{return}) < 1$.

For $d \ge 3$, consider the projection of the random walk into the first three coordinates. The random walk does not return to the origin if the first three coordinates does not.

Example: Let $\{S_n : n \ge 0\}$ be a random walk which moves up with probability p at each step. Show that $P(S_n = 0 \text{ i.o.}) = 0$ if $p \ne \frac{1}{2}$.

$$P(S_{2n} = 0) = \binom{2n}{n}\{p(1-p)\}^n \sim [4p(1-p)]^n\frac{1}{\sqrt{\pi n}}$$

where $[4p(1-p)] < 1$ for $p \ne \frac{1}{2}$, and this implies that this sum is less than ∞. Therefore a direct application of Borel-Cantelli lemma (1) shows that $P(S_n = 0 \text{ i.o.}) = 0$.