MAT235: Discussion 7

1 Characteristic Function

Let $\phi(t) = E(e^{itX})$ where $i = \sqrt{-1}$ denote the characteristic function.

1.1 Some Properties

1. ϕ is non-negative definite, which is to say that $\sum_{j,k} \phi(t_j - t_k) z_j \bar{z}_k \geq 0$ for all real $t_1, t_2 \cdots, t_n$ and complex $z_1, z_2, \cdots, z_n$.
   Proof.
$$\begin{aligned}\sum_{j,k} \phi(t_j - t_k) z_j \bar{z}_k &= \sum_{j,k} \int [z_j \exp(it_j x)][\bar{z}_k \exp(-it_k x)] dF \\ &= E\Big(|\sum_j z_j \exp(it_j X)|^2\Big) \geq 0.\end{aligned}$$

2. If $\phi^{(k)}(0)$ exists then
$$\begin{cases} E|X^k| < \infty & \text{if } k \text{ is even} \\ E|X^{k-1}| < \infty & \text{if } k \text{ is odd} \end{cases}$$

3. If $E|X^k| < \infty$ then
$$\phi(t) = \sum_{j=0}^{k} \frac{E(X^j)}{j!} (it)^j + o(t^k),$$
   and so $\phi^{(k)}(0) = i^k E(X^k)$.
   Proof of 1, 3 is essentially Taylor's theorem for a function of a complex variable.

Example 1. Bernoulli distribution. If X is Bernoulli with parameter p then

$$\phi(t) = E(e^{itX}) = e^{it\cdot 0} q + e^{it \cdot 1} p = q + pe^{it}.$$

Example 2. Binomial distribution. If X is $Bin(n,p)$, then it is the sum of n Bernoulli random variables. Hence

$$\phi(t) = E(e^{itX}) = (q + pe^{it})^n.$$

Example 3. It is known that if X and Y are independent, then $\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$. But is the converse true in general.
Consider the Cauchy distribution with characteristic function $\phi(t) = e^{-|t|}$. Let X have Cauchy distribution and set $Y = X$. Then $\phi_{X+Y}(t) = \phi(2t) = e^{-2|t|} = \phi_X(t)\phi_Y(t)$.

Remark. If a distribution F is given, then the corresponding moments $m_k(F) = \int x^k dF(x), k = 1, 2, \cdots$ whenever these integrals exist. Is the converse true: does the collection of moments $(m_k(F) : k = 1, 2, \cdots)$ specify F uniquely? The answer is no.

Example 4. Log-normal distribution. Let X be $N(0,1)$, and let $Y = e^X$; Y is said to have the log-normal distribution. Show that the density function of Y is

$$f(x) = \frac{1}{x\sqrt{2\pi}} \exp -\frac{1}{2}(\log x)^2, \quad x > 0.$$

For $|a| \le 1$, define $f_a(x) = \{1 + a\sin(2\pi \log x)\}f(x)$. Show that f_a is a density function with finite moments of all (positive) orders, none of which depends on the value of a. The family $\{f_a : |a| \le 1\}$ contains density functions which are not specified by their moments.

Solution. (i) $P(Y \le y) = P(X \le \log y) = \Phi(\log y)$ for $y > 0$, where Φ is the c.d.f. of $N(0, 1)$. The density function of Y follows by differentiating.

(ii) Notice that $f_a(x) \ge 0$ if $|a| \le 1$, and

$$\int_0^\infty a\sin(2\pi \log x)\frac{1}{x\sqrt{2\pi}}\exp(-\frac{1}{2}(\log x)^2)dx = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}a\sin(2\pi y)\exp(-\frac{1}{2}y^2)dy = 0$$

since sine is an odd function. Therefore $\int_{-\infty}^\infty f_a(x)dx = 1$, so that each such f_a is a density function.

(iii) For any positive integer k, the k-th moment of f_a is $\int_{-\infty}^\infty x^k f(x)dx + I_a(k)$ where

$$I_a(k) = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}a\sin(2\pi y)e^{ky-\frac{1}{2}y^2}dy = 0$$

since the integrand is an odd function of $y - k$. It follow that each f_a has the same moments as f.

Under what condition on F is it the case that the moments uniquely specify the distribution. One of the simplest sufficient condition is that the moment generating function of F is finite in some neighborhood of the point t. Remember the moment generating function is defined as $M(t) = E(e^{tX})$, it is clear that characteristic function is closely related to moment generating function.

Theorem. Let $M(t) = E(e^{tX}), t \in R$, and $\phi(t) = E(e^{itX}), t \in C$ be the moment generating function and characteristic function of X respectively. For any $a > 0$, the following conditions are equivalent.

(a) $|M(t)| < \infty$ for $|t| < a$.

(b) ϕ is analytic on the strip $|Im(z)| < a$.

(c) The moments $m_k = E(X^k)$ exist for $k = 1, 2, \cdots$ and satisfy $\limsup_{k\to\infty}\{|m_k|/k!\}^{1/k} \le a^{-1}$.

If any of these conditions hold for $a > 0$, the power series expansion for $M(t)$ may be extended analytically to the strip $|Im(z) < a|$, resulting in a function M with the property that $\phi(t) = M(it)$.