MAT235: Discussion 8

Modes of convergence
Definition: Let $X_1, X_2, \cdots$ be random variables on some probability space (Ω, F, P).

1. $X_n \to_{a.s.} X$ almost surely if $\{\omega \in \Omega : X(\omega) \to X(\omega)$ as $n \to \infty\}$ is an event of probability 1.
2. $X_n \to_r X$ in r-th mean where $r \geq 1$ if $E|X_n^r| < \infty$ for all n and $E(|X_n - X|^r) \to 0$ as $n \to \infty$.
3. $X_n \to_P X$ in probability if $P(|X_n - X| > \epsilon) \to 0$ as $n \to \infty$ for all $\epsilon > 0$.
4. $X_n \to_D X$ in distribution if $P(X_n \leq x) \to P(X \leq x)$ as $n \to \infty$ for all points x at which the function $F_X(x)$ is continuous.

Lemma 1: If $r > s \geq 1$ and $X_n \to_r X$ then $X_n \to_s X$.
Proof. First apply Cauchy-Schwarz inequality to $|X|^{\frac{1}{2}(b-a)}$ and $|X|^{\frac{1}{2}(b+a)}$, where $0 < a \leq b$, to obtain $(E|Z^b|)^2 \leq E|Z|^{(b-a)}E|Z|^{(b+a)}$. Let $g(p) = \log E|Z^p|$, we have $2g(b) \leq g(b-a) + g(b+a)$, $g(p)$ is a convex function. Therefore $g(x)/x$ is non-decreasing in x, and hence $g(r)/r \geq g(s)/s$ if $0 < s \leq r$. The claim then follows.
To see that the converse fails, define an independent sequence X_n by

$$X_n = \begin{cases} n & \text{with probability } n^{-\frac{1}{2}(r+s)} \\ 0 & \text{with probability } 1 - n^{-\frac{1}{2}(r+s)}. \end{cases}$$

It is easy to check that $E|X_n^s| = n^{\frac{1}{2}(s-r)} \to 0$ but $E|X_n^r| = n^{\frac{1}{2}(r-s)} \to \infty$.

Lemma 2: If $X_n \to_1 X$ then $X_n \to_P X$.
Proof. By Markov's inequality:

$$P(|X_n - X| > \epsilon) \leq \frac{E|X_n - X|}{\epsilon} \quad \text{for all } \epsilon > 0.$$

To see the converse fails, define an independent sequence $\{X_n\}$ by

$$X_n = \begin{cases} n^3 & \text{with probability } n^{-2} \\ 0 & \text{with probability } 1 - n^{-2}. \end{cases}$$

Then $P(|X_n| > \epsilon) = n^{-2} \to 0$ but $E|X_n| = n \to \infty$.

Lemma 3: $(X_n \to_P X) \Rightarrow (X_n \to_D X)$.
Counter example: Let X be a $Bernoulli(\frac{1}{2})$. Let $X_1, X_2, \cdots$ be identical random variables given by $X_n = X$ for all n. Let $Y = 1 - X$, then Y and X have the same distribution. Hence $X_n \to_D Y$. However clearly X_n can not converge to Y in probability since $|X_n - Y| = 1$ for all n.

Lemma 4: $(X_n \to_{a.s.} X) \Rightarrow (X_n \to_P X)$.
Counter example: Define $\{X_n\}$ to be a sequence of random variables on $(0,1)$ with Borel measure as the probability measure. The sequence is defined in following steps.

- Step 1: $X_1 = 1$ on (0,1).
- Step 2: $X_{21} = 1$ on (0,1/2], $X_{22} = 0$ on (1/2,1); $X_{22} = 0$ on (0,1/2], $X_{22} = 1$ on (1/2,1).
- Step k: $X_{kj} = 1$ on (j/k,(j+1)/k], $X_{kj} = 0$ other wise where $j = 1, 2, \cdots, k$.

It is too see that X_n converges to 0 in probability but not almost surely. In fact X_n converges no where on (0,1).

Lemma 5: There exist sequences which:
(a) converge almost surely but not in mean. Consider the example in Lemma 2.
(b) converge in mean but not almost surely. Consider the example in Lemma 4.

Example. Show that $X_n \to_{a.s.} X$ whenever $\Sigma_n E(|X_n - X|^r) < \infty$ for some $r > 0$.
Proof. We have by Markov's inequality that

$$\sum_n P(|X_n - X| > \epsilon) \leq \sum_n \frac{E|X_n - X|^r}{\epsilon^r} < \infty$$

for $\epsilon > 0$, so that $X_n \to_{a.s.} X$ by first Borel-Cantelli lemma.

Example. Show that $X_n \to_P 0$ if and only if $E(\frac{|X_n|}{1+|X_n|}) \to 0$ as $n \to \infty$.
Proof. Note that $g(u) = u/(1+u)$ is an increasing function on $[0, \infty)$. Therefore for $\epsilon > 0$,

$$P(|X_n| > \epsilon) = P(\frac{|X_n|}{1+|X_n|} > \frac{\epsilon}{1+\epsilon}) \leq \frac{1+\epsilon}{\epsilon} E(\frac{|X_n|}{1+|X_n|})$$

by Markov's inequality. Hence $E(\frac{|X_n|}{1+|X_n|}) \to 0$ implies $P(|X_n| > \epsilon) \to 0$.
Suppose conversely that $X_n \to_P 0$. Then

$$E(\frac{|X_n|}{1+|X_n|}) \leq \frac{\epsilon}{1+\epsilon} P(|X_n| \leq \epsilon) + P(|X_n| > \epsilon) \to \frac{\epsilon}{1+\epsilon}$$

as $n \to \infty$, for $\epsilon > 0$. However ϵ is arbitrary, and hence the expectation has limit 0.

Example: Let X_n and Y_m be independent random variables having Poisson distribution with parameters n and m respectively. Show that as $m, n \to \infty$

$$\frac{(X_n - n) - (Y_m - m)}{\sqrt{X_n + Y_m}} \to_D N(0,1)$$

. Proof. The characteristic function ϕ_{mn} of

$$U_{mn} = \frac{(X_n - n) - (Y_m - m)}{\sqrt{m+n}}$$

satisfies

$$\log \phi_{mn}(t) = n(e^{it/\sqrt{m+n}} - 1) + m(e^{-it/\sqrt{m+n}} - 1) + \frac{(m-n)it}{\sqrt{m+n}} \to -\frac{t^2}{2}$$

an $m, n \to \infty$ by Taylor's expansion. This implies that $U_{mn} \to_D N(0,1)$. Now $X_n + Y_m$ is Poisson-distributed with parameter $m + n$, and therefore

$$V_{mn} = \sqrt{\frac{X_n + Y_m}{m+n}} \to_P 1 \quad \text{as } m, n \to \infty$$

by law of large numbers and the continuity of square root function. Then Slutsky's theorem shows that $U_{mn}/V_{mn} \to_D N(0,1)$.