MAT235: Discussion 8

Modes of convergence

Definition: Let $X_1, X_2, \cdots$ be random variables on some probability space $(\Omega, F, P)$.

1. $X_n \to_{a.s.} X$ almost surely if $\{\omega \in \Omega : X(\omega) \to X(\omega) \text{ as } n \to \infty\}$ is an event of probability 1.
2. $X_n \to_r X$ in $r$-th mean where $r \geq 1$ if $E[X_n^r] < \infty$ for all $n$ and $E(|X_n - X|^r) \to 0$ as $n \to \infty$.
3. $X_n \to_P X$ in probability if $P(|X_n - X| > \epsilon) \to 0$ as $n \to \infty$ for all $\epsilon > 0$.
4. $X_n \to_D X$ in distribution if $P(X_n \leq x) \to P(X \leq x)$ as $n \to \infty$ for all points $x$ at which the function $F_X(x)$ is continuous.

Lemma 1: If $r > s \geq 1$ and $X_n \to_r X$ then $X_n \to_s X$.

Proof. First apply Cauchy-Schwarz inequality to $|X|^\frac{1}{2} + a \text{ and } |X|^\frac{1}{2} + b$, where $0 < a \leq b$, to obtain $(E[Z^b])^2 \leq E[Z^{b-a}]E[Z^{b+a}]$. Let $g(p) = \log |Z|^p$, we have $2g(b) \leq g(b-a) + g(b+a)$, $g(p)$ is a convex function. Therefore $g(x)/x$ is non-decreasing in $x$, and hence $g(r)/r \geq g(s)/s$ if $0 < s \leq r$. The claim then follows.

To see that the converse fails, define an independent sequence $X_n$ by

$$X_n = \begin{cases} n & \text{with probability } n^{-\frac{1}{2}(r+s)} \\ 0 & \text{with probability } 1 - n^{-\frac{1}{2}(r+s)}. \end{cases}$$

It is easy to check that $E[|X_n|^s] = n^{-\frac{1}{2}(s-r)} \to 0$ but $E[|X_n|^r] = n^{-\frac{1}{2}(r-s)} \to \infty$.

Lemma 2: If $X_n \to_1 X$ then $X_n \to_P X$.

Proof. By Markov’s inequality:

$$P(|X_n - X| > \epsilon) \leq \frac{E[|X_n - X|]}{\epsilon} \quad \text{for all } \epsilon > 0.$$

To see the converse fails, define an independent sequence $\{X_n\}$ by

$$X_n = \begin{cases} n^3 & \text{with probability } n^{-2} \\ 0 & \text{with probability } 1 - n^{-2}. \end{cases}$$

Then $P(|X_n| > \epsilon) = n^{-2} \to 0$ but $E[|X_n|] = n \to \infty$.

Lemma 3: $(X_n \to_P X) \Rightarrow (X_n \to_D X)$.

Counter example: Let $X$ be a $Bernoulli(\frac{1}{2})$. Let $X_1, X_2, \cdots$ be identical random variables given by $X_n = X$ for all $n$. Let $Y = 1 - X$, then $Y$ and $X$ have the same distribution. Hence $X_n \to_D Y$. However clearly $X_n$ can not converge to $Y$ in probability since $|X_n - Y| = 1$ for all $n$.

Lemma 4: $(X_n \to_{a.s.} X) \Rightarrow (X_n \to_P X)$.

Counter example: Define $\{X_n\}$ to be a sequence of random variables on $(0,1)$ with Borel measure as the probability measure. The sequence is defined in following steps.

- Step 1: $X_1 = 1$ on $(0,1)$.
- Step 2: $X_{21} = 1$ on $(0,1/2]$, $X_{22} = 0$ on $(1/2,1)$; $X_{22} = 0$ on $(0,1/2]$, $X_{22} = 1$ on $(1/2,1)$.
- Step $k$: $X_{kj} = 1$ on $(j/k,(j+1)/k]$, $X_{kj} = 0$ other wise where $j = 1,2,\cdots, k$. 

It is too see that $X_n$ converges to 0 in probability but not almost surely. In fact $X_n$ converges no where on $(0,1)$.

**Lemma 5.** There exist sequences which:
(a) converge almost surely but not in mean. Consider the example in Lemma 2.
(b) converge in mean but not almost surely. Consider the example in Lemma 4.

**Example.** Show that $X_n \to a.s. X$ whenever $\sum_n E(|X_n - X|^r) < \infty$ for some $r > 0$. 
Proof. We have by Markov’s inequality that
$$
\sum_n P(|X_n - X| > \epsilon) \leq \sum_n \frac{E|X_n - X|^r}{\epsilon^r} < \infty
$$
for $\epsilon > 0$, so that $X_n \to a.s. X$ by first Borel-Cantelli lemma.

**Example.** Show that $X_n \to P 0$ if and only if $E(|X_n| 1+|X_n|) \to 0$ as $n \to \infty$.

Proof. Note that $g(u) = u/(1+u)$ is an increasing function on $[0, \infty)$. Therefore for $\epsilon > 0,$
$$
P(|X_n| > \epsilon) = P\left( \frac{|X_n|}{1 + |X_n|} > \frac{\epsilon}{1 + \epsilon} \right) \leq \frac{1 + \epsilon}{\epsilon} E\left( \frac{|X_n|}{1 + |X_n|} \right)
$$
by Markov’s inequality. Hence $E\left( \frac{|X_n|}{1 + |X_n|} \right) \to 0$ implies $P(|X_n| > \epsilon) \to 0$.

Suppose conversely that $X_n \to P 0$. Then
$$
E\left( \frac{|X_n|}{1 + |X_n|} \right) \leq \frac{\epsilon}{1 + \epsilon} P(|X_n| \leq \epsilon) + P(|X_n| > \epsilon) \to \frac{\epsilon}{1 + \epsilon}
$$
as $n \to \infty$, for $\epsilon > 0$. However $\epsilon$ is arbitrary, and hence the expectation has limit 0.

**Example:** Let $X_n$ and $Y_n$ be independent random variables having Poisson distribution with parameters $n$ and $m$ respectively. Show that as $m, n \to \infty$
$$
\frac{(X_n - n) - (Y_m - m)}{\sqrt{X_n + Y_m}} \to_D N(0, 1)
$$

. Proof. The characteristic function $\phi_{mn}$ of
$$
U_{mn} = \frac{(X_n - n) - (Y_m - m)}{\sqrt{m + n}}
$$
satisfies
$$
\log \phi_{mn}(t) = n(e^{it/\sqrt{m+n}} - 1) + m(e^{-it/\sqrt{m+n}} - 1) + \frac{(m-n)it}{\sqrt{m+n}} \to \frac{t^2}{2}
$$
an $m, n \to \infty$ by Taylor’s expansion. This implies that $U_{mn} \to_D N(0, 1)$. Now $X_n + Y_m$ is Poisson-distributed with parameter $m + n$, and therefore
$$
V_{mn} = \sqrt{\frac{X_n + Y_m}{m + n}} \to_P 1 \quad \text{as } m, n \to \infty
$$
by law of large numbers and the continuity of square root function. Then Slutsky’s theorem shows that $U_{mn}/V_{mn} \to_D N(0, 1)$. 

2