Homework Set No. 1 – Probability Theory (235A), Fall 2013

Due: 10/7/13 at discussion section

1. (The total probability formula) If $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space and $A, B \in \mathcal{F}$ are events such that $\mathbf{P}(B) \neq 0$, the **conditional probability of** A **given** B is denoted $\mathbf{P}(A|B)$ and defined by

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)}.$$

Prove the **total probability formula:** if $A, B_1, B_2, \ldots, B_k \in \mathcal{F}$ such that Ω is the disjoint union of $B_1, \ldots, B_k$ and $\mathbf{P}(B_i) \neq 0$ for $1 \leq i \leq k$, then

$$\mathbf{P}(A) = \sum_{i=1}^{k} \mathbf{P}(B_i)\mathbf{P}(A|B_i). \tag{TPF}$$

2. (Pólya's urn) An urn initially contains one white ball and one black ball. At each step of the experiment, a ball is drawn at random from the urn, then put back and another ball *of the same color* is added. Prove that the number of white balls that are in the urn after N steps is a uniform random number in $\{1, 2, \ldots, N+1\}$. That is, the event that the number of white balls after step N is equal to k has probability $1/(N+1)$ for each $1 \leq k \leq N+1$. (Note: The idea is to use (TPF), but there is no need to be too formal about constructing the relevant probability space - you can assume an intuitive notion of probabilities.)

3. (Randomized coordination) A cluster of N computers need to communicate with a central server. At any time $t = 0, 1, 2, \ldots$, each computer can decide to transmit, or not to transmit, a data packet to the server. However, the server has very limited computing capacity: it can process the data if and only if *precisely one computer* has sent a packet; if no data was sent, or if more than one computer sent a packet, the data (and time and effort spent sending it) is lost.

The computers cannot communicate directly with each other to coordinate the order in which they send their packets. Show that they can nonetheless adopt the following randomized strategy to ensure a relatively efficient rate of data transfer: for some value $0 < p < 1$, each computer at each time t tosses a coin with bias p, independently of all other computers, and sends its packet if the coin comes up "heads". Find the value of p (as a function of N) that would lead to the maximal rate of success (i.e., probability at each time t for the server to process data), and find that rate of success as a function of N and in the limit when $N \to \infty$.

4. (The inclusion-exclusion principle) If $A_1, \ldots, A_n$ are events in a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, prove the formula

$$\mathbf{P}\left(\bigcup_{k=1}^{n} A_k\right) = s_1 - s_2 + s_3 - s_4 + s_5 - \ldots + (-1)^{n-1} s_n,$$

where

$$\begin{aligned}
s_1 &= \mathbf{P}(A_1) + \mathbf{P}(A_2) + \ldots + \mathbf{P}(A_n) = \sum_{k=1}^{n} \mathbf{P}(A_k), \\
s_2 &= \sum_{1 \le k_1 < k_2 \le n} \mathbf{P}(A_{k_1} \cap A_{k_2}), \\
s_3 &= \sum_{1 \le k_1 < k_2 < k_3 \le n} \mathbf{P}(A_{k_1} \cap A_{k_2} \cap A_{k_3}), \\
&\vdots \\
s_d &= \sum_{1 \le k_1 < \ldots < k_d \le n} \mathbf{P}(A_{k_1} \cap A_{k_2} \cap \ldots \cap A_{k_d}), \\
&\vdots \\
s_n &= \mathbf{P}(A_1 \cap A_2 \cap \ldots \cap A_n).
\end{aligned}$$

5. (The derangement problem) N letters addressed to different people are inserted at random into N envelopes that are labelled with the names and addresses of the N recipients, such that all $N!$ possible matchings between the letters and envelopes are equally likely. What is the probability of the event that no letter will arrive at its intended destination? As an application of the formula in problem 3 above, compute this probability for any N, and in the limit when $N \to \infty$.

6. (Poisson splitting) Let Z be a random variable with the Poi(λ) distribution. For example, Z could represent the number of frog specimens observed by a biologist in a patch of jungle on a certain day, where the parameter λ corresponds to the number of frogs seen on an average day. Assume that the events counted by Z are split into two types, where the splitting happens independently at random for each event; for example, frogs could be classified as belonging to one of two species. Formally, let $X_1, X_2, X_3, \ldots$ denote an infinite sequence of i.i.d. (independent and identically distributed) random variables with the $\mathrm{Ber}(p)$ distribution (i.e., $\mathbf{P}(X_n = 1) = 1 - \mathbf{P}(X_n = 0) = p$), where $0 < p < 1$ is some parameter. Define

$$W_1 = \sum_{n=0}^{Z} X_n,$$
$$W_2 = \sum_{n=0}^{Z} (1 - X_n),$$

so that $Z = W_1 + W_2$, and in the example W_1 would represent the number of frogs belonging to the first species and W_2 the number of frogs from the second species. (The parameter p is the fraction of frogs belonging to the first species.)

Prove that W_1, W_2 are also Poisson random variables with distributions $W_1 \sim \mathrm{Poi}(\lambda p)$, $W_2 \sim \mathrm{Poi}(\lambda(1-p))$, and they are independent of each other.