Homework Set No. 2 – Probability Theory (235A), Fall 2013

Due: 10/14/13 at discussion section

1. (Transformation of a random variable) Let X be a random variable with distribution function F_X and piecewise continuous density function f_X. Let $[a,b] \subset \mathbb{R}$ be an interval (possibly infinite) such that

$$\mathbf{P}(X \in [a,b]) = 1,$$

and let $g : [a,b] \to \mathbb{R}$ be a monotone (strictly) increasing and differentiable function. Prove that the random variable $Y = g(X)$ (this is the function on Ω defined by $Y(\omega) = g(X(\omega))$, in other words the composition of the two functions g and X) has density function

$$f_Y(x) = \begin{cases} \frac{f_X(g^{-1}(x))}{g'(g^{-1}(x))} & x \in (g(a), g(b)), \\ 0 & \text{otherwise.} \end{cases}$$

2. (Sampling an exponential random variable) If $\lambda > 0$, we say that a random variable has the exponential distribution with parameter λ if

$$F_X(x) = \begin{cases} 0 & x < 0, \\ 1 - e^{-\lambda x} & x \geq 0, \end{cases}$$

and denote this $X \sim \mathrm{Exp}(\lambda)$. Find an algorithm to produce a random variable with $\mathrm{Exp}(\lambda)$ distribution using a random number generator that produces uniform random numbers in $(0,1)$. In other words, if $U \sim U(0,1)$, find a function $g : (0,1) \to \mathbb{R}$ such that the random variable $X = g(U)$ has distribution $\mathrm{Exp}(\lambda)$.

3. (The lack of memory property) (a) We say that a non-negative random variable $X \geq 0$ has the **lack of memory property** if it satisfies that

$$\mathbf{P}(X \geq t \mid X \geq s) = \mathbf{P}(X \geq t - s) \qquad \text{for all } 0 < s < t.$$

Show that exponential random variables have the lack of memory property.

(b) Prove that any non-negative random variable that has the lack of memory property has the exponential distribution with some parameter $\lambda > 0$. (This is easier if one assumes that the function $G(x) = \mathbf{P}(X \geq x)$ is differentiable on $[0, \infty)$, so you can make this assumption if you fail to find a more general argument).

3. (Quantile functions) Let F be the distribution function

$$F(x) = \begin{cases} 0 & x < 0, \\ \frac{1}{3} + \frac{1}{6}x & 0 \leq x < 1, \\ \frac{1}{2} & 1 \leq x < 2, \\ 1 - \frac{1}{4}e^{2-x} & x \geq 2. \end{cases}$$

Compute the lower and upper quantile functions of F, defined by

$$\begin{aligned} X_*(p) &= \sup\{x : F(x) < p\}, \\ X^*(p) &= \inf\{x : F(x) > p\}, \end{aligned} \qquad (0 < p < 1).$$

(It may be helpful to start by plotting F on paper and then figure out the quantiles visually. However, the final answer should be spelled out in precise formulas.)

4. (An interesting distribution) A drunken archer shoots at a target hanging on a wall 1 unit of distance away. Since he is drunk, his arrow ends up going in a random direction at an angle Θ chosen uniformly in $(-\pi/2, \pi/2)$ (an angle of 0 means he will hit the target precisely) until it hits the wall. Ignoring gravity and the third dimension, compute the distribution function, and density function if it exists, of the random distance X from the hitting point of the arrow to the target.

5. (An insufficient condition) Let $\Omega = \{1, 2, 3, 4\}$, and let $\mathcal{A} = \{\{1, 2\}, \{1, 3\}\}$.

(a) Verify that $\mathcal{F} = \sigma(\mathcal{A})$ (the σ-algebra generated by $\mathcal{A}$) is the power set $\mathcal{P}(\Omega)$.

(b) Show that there are two probability measures P_1 and P_2 on $(\Omega, \mathcal{F})$ such that $P_1(E) = P_2(E)$ for any $E \in \mathcal{A}$, but $P_1 \neq P_2$.

Note. this problem demonstrates that the fact that two probability measures coincide on a generating family of a σ-algebra does not guarantee that they are equal. However, this implication is correct if one adds the assumption that the generating family $\mathcal{A}$ is closed under finite intersections.sufficient; see Theorem A.1.5 in Appendix A of Rick Durrett's book *Probability: Theory and Examples, 4th Ed.*.

6. (Composition of measurable functions) Let $(\Omega_1, \mathcal{F}_1), (\Omega_2, \mathcal{F}_2), (\Omega_3, \mathcal{F}_3)$ be measurable spaces. Show that if $f : (\Omega_1, \mathcal{F}_1) \to (\Omega_2, \mathcal{F}_2)$ and $g : (\Omega_2, \mathcal{F}_2) \to (\Omega_3, \mathcal{F}_3)$ are measurable functions then their composition $g \circ f : (\Omega_1, \mathcal{F}_1) \to (\Omega_3, \mathcal{F}_3)$ is also measurable.

7. (Operations on random variables) Show that if X, Y are random variables on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ then the following are also random variables:

(a) $|X|$

(b) $X + Y$

(c) $X \cdot Y$

(d) $\max(X, Y)$

(e) $\min(X, Y)$

Hint. For example for (b), the idea is to represent the set $\{X + Y < t\} \subset \Omega$ in terms of legal operations (within a σ-algebra) on sets known to be events such as $\{X \in (a, b)\}$, $\{Y \in (a, b)\}$.