Homework Set No. 3 – Probability Theory (235A), Fall 2013

Due: 10/21/13 at discussion section

1. Let X be an exponential r.v. with parameter λ, i.e., $F_X(x) = (1 - e^{-\lambda x})1_{[0,\infty)}(x)$. Define random variables

$$\begin{aligned} Y &= \lfloor X \rfloor := \sup\{n \in \mathbb{Z} : n \le x\} \qquad &\text{(“the integer part of } X\text{”)}, \\ Z &= \{X\} := X - \lfloor X \rfloor \qquad &\text{(“the fractional part of } X\text{”)}. \end{aligned}$$

(a) Compute the (1-dimensional) distributions of Y and Z (in the case of Y, since it's a discrete random variable it is most convenient to describe the distribution by giving the individual probabilities $\mathbf{P}(Y = n), n = 0, 1, 2, \ldots$; for Z one should compute either the distribution function or density function).

(b) Show that Y and Z are independent. (Hint: Check that $\mathbf{P}(Y = n, Z \le t) = \mathbf{P}(Y = n)\mathbf{P}(Z \le t)$ for all n and t.)

2. Use the convolution formula

$$f_{X+Y}(t) = \int_{-\infty}^{\infty} f_X(x) f_Y(t - x)\, dx$$

for the density of a sum of independent random variables to compute the distribution of $X + Y$ when X and Y are independent r.v.'s with the following pairs of distributions:

(a) $X \sim U[0,1],\ Y \sim U[0,2]$.

(b) $X \sim \mathrm{Exp}(1),\ Y \sim \mathrm{Exp}(1)$.

(c) $X \sim \mathrm{Exp}(1),\ -Y \sim \mathrm{Exp}(1)$.

(d) $X \sim \mathrm{Exp}(1),\ Y \sim U[0,1]$.

3. Prove that if X is a random variable that is independent of itself, then X is a.s. constant, i.e., there is a constant $c \in \mathbb{R}$ such that $\mathbf{P}(X = c) = 1$.

4. (a) Show that if X, Y are independent random variables with the standard normal distribution, and $a, b \in \mathbb{R}$ are numbers such that $a^2 + b^2 = 1$, then $aX + bY \sim N(0, 1)$.

Hints. Here are two suggested methods. The first possibility is to use the polar decomposition (R, Θ) of the random vector (X, Y), defined by the relations $X = R\cos\Theta, Y = R\sin\Theta$. The second possible approach is to consider $U = aX + bY$ as one coordinate of the vector

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix},$$

and to compute the joint density of (U, V) from the joint density of (X, Y) using the transformation formula for 2-dimensional densities (developed in the discussion section), noting that the matrix $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ is a rotation matrix.

(b) Use part (a) to prove that if $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ are independent then $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$. (A useful fact to be aware of is that $Z \sim N(m, s^2) \iff \frac{Z-m}{s} \sim N(0, 1)$.)

(c) Optionally, prove the result of part (b) directly (by replacing X and Y with $X' = X - \mu_1$ and $Y' = Y - \mu_2$ it is enough to consider the special case $\mu_1 = \mu_2 = 0$) by computing the convolution of two normal densities.

5. If X, Y are independent random variables with the standard normal distribution $N(0, 1)$, show that $X^2 - Y^2$, $2XY$ are equal in distribution.

6. Let X, Y be i.i.d. random variables with distribution function $F = F_X = F_Y$.

(a) Show that the pair of random variables $U = \min(X, Y)$, $V = \max(X, Y)$ have joint distribution function

$$F_{U,V}(u, v) = \begin{cases} F(u)(2F(v) - F(u)) & \text{if } u \le v, \\ F(v)^2 & \text{if } u > v. \end{cases}$$

(b) In the case when X, Y also have density f, use the result above to show that U, V have joint density

$$f_{U,V}(u, v) = \begin{cases} 2f(u)f(v) & \text{if } u < v, \\ 0 & \text{otherwise.} \end{cases}$$

(c) Apply the above formula in the case $X, Y \sim U[0,1]$ to get the joint density of U, V in this case. Then compute the marginal (1-dimensional) densities of U, V using the relations

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,dv,$$
$$f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,du,$$

and use them to identify which special distribution family each of the r.v.s U, V and belongs to, and with what parameters.

7. Let $\Gamma(t)$ denote the *Euler gamma function*, defined by

$$\Gamma(t) = \int_0^{\infty} e^{-x} x^{t-1}\,dx, \qquad (t > 0).$$

(a) Show that the special value $\Gamma(1/2) = \sqrt{\pi}$ of the gamma function is equivalent to the integral evaluation $\sqrt{2\pi} = \int_{-\infty}^{\infty} e^{-x^2/2}\,dx$ (which is equivalent to the standard normal density being a density function).

(b) Prove that the Euler gamma function satisfies for all $t > 0$ the identity

$$\Gamma(t+1) = t\,\Gamma(t).$$

(This identity immediately implies the fact that $\Gamma(n+1) = n!$ for integer $n \geq 0$.)

(c) Find a formula for the values of $\Gamma(\cdot)$ at half-integers, that is,

$$\Gamma\left(n + \tfrac{1}{2}\right) = ?, \qquad (n \geq 0).$$

(d) Read the Wikipedia article on the Euler gamma function to learn about some of its other very interesting properties and its importance in mathematics.