Homework Set No. 4 – Probability Theory (235A), Fall 2013

Due: 10/28/13 at discussion section

1. Compute $\mathbf{E}(X)$ and $\mathrm{Var}(X)$ when X is a random variable having each of the following distributions:

1. $X \sim \mathrm{Binomial}(n, p)$.
2. $X \sim \mathrm{Poisson}(\lambda)$.
3. $X \sim \mathrm{Geom}(p)$.
4. $X \sim U\{1, 2, \ldots, n\}$ (the discrete uniform distribution on $\{1, 2, \ldots, n\}$).
5. $X \sim U(a, b)$ (the uniform distribution on the interval (a, b)).
6. $X \sim \mathrm{Exp}(\lambda)$
7. $X \sim \mathrm{Gamma}(\alpha, \lambda)$
8. $X \sim \mathrm{Beta}(\alpha, \beta)$

2. If X is a random variable satisfying $a \le X \le b$, prove that

$$\mathrm{Var}(X) \le \frac{(b-a)^2}{4},$$

and identify when equality holds.

3. A function $\varphi : (a, b) \to \mathbb{R}$ is called **convex** if for any $x, y \in (a, b)$ and $\alpha \in [0, 1]$ we have

$$\varphi(\alpha x + (1-\alpha)y) \le \alpha\varphi(x) + (1-\alpha)\varphi(y).$$

(a) Prove that an equivalent condition for φ to be convex is that for any $x < z < y$ in (a, b) we have

$$\frac{\varphi(z) - \varphi(x)}{z - x} \le \frac{\varphi(y) - \varphi(z)}{y - z}.$$

Deduce using the mean value theorem that if φ is twice continuously differentiable and satisfies $\varphi'' \geq 0$ then it is convex.

(b) Prove **Jensen's inequality**, which says that if X is a random variable such that $\mathbf{P}(X \in (a,b)) = 1$ and $\varphi : (a,b) \to \mathbb{R}$ is convex, then

$$\varphi(\mathbf{E}X) \leq \mathbf{E}(\varphi(X)).$$

Hint. Start by proving the following property of a convex function: If φ is convex then at any point $x_0 \in (a,b)$, φ has a **supporting line**, that is, a linear function $y(x) = ax + b$ such that $y(x_0) = \varphi(x_0)$ and such that $\varphi(x) \geq y(x)$ for all $x \in (a,b)$ (to prove its existence, use the characterization of convexity from part (a) to show that the left-sided derivative of φ at x_0 is less than or equal to the right-sided derivative at x_0; the supporting line is a line passing through the point $(x_0, \varphi(x_0))$ whose slope lies between these two numbers). Now take the supporting line function at $x_0 = \mathbf{E}X$ and make appropriate use of the monotonicity property of the expectation.

4. Let $(A_n)_{n=1}^{\infty}$ be a sequence of events in a probability space. Show that

$$1_{\limsup A_n} = \limsup_n 1_{A_n}.$$

(The lim-sup on the left refers to the lim-sup operation on events; on the right it refers to the lim-sup of a sequence of functions; the identity is an identity of real-valued functions on Ω, i.e., should be satisfied for each individual point $\omega \in \Omega$ in the sample space). Similarly, show (either separately or by relying on the first claim) that

$$1_{\liminf A_n} = \liminf_n 1_{A_n}.$$

5. Let U be a uniform random variable in $(0,1)$. For each $n \geq 1$ define an event A_n by

$$A_n = \{U < 1/n\}.$$

Note that $\sum_{n=1}^{\infty} \mathbf{P}(A_n) = \infty$. However, compute $\mathbf{P}(A_n \text{ i.o.})$ and show that the conclusion of the second Borel-Cantelli lemma does not hold (of course, one of the assumptions of the lemma also doesn't hold, so there's no contradiction).

6. If P, Q are two probability measures on a measurable space $(\Omega, \mathcal{F})$, we say that P **is absolutely continuous with respect to** Q, and denote this $P << Q$, if for any $A \in \mathcal{F}$, if $Q(A) = 0$ then $P(A) = 0$.

Prove that $P << Q$ if and only if for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $A \in \mathcal{F}$ and $Q(A) < \delta$ then $P(A) < \epsilon$.

Hint. The first Borel-Cantelli lemma makes an interesting appearance here.