Homework Set No. 5 – Probability Theory (235A), Fall 2013

Due: 11/4/13 at discussion section

To get full credit, solve problems 1–3 and at least one of problems 4–6.

1. If $X \ge 0$ is a nonnegative r.v. with distribution function F, show that

$$\mathbf{E}(X) = \int_0^\infty \mathbf{P}(X \ge x)\,dx.$$

2. (a) Prove that if $X_1, X_2, \ldots,$ is a sequence of independent and identically distributed ("i.i.d.") r.v.'s, then

$$\mathbf{P}(|X_n| \ge n \text{ i.o.}) = \begin{cases} 0 & \text{if } \mathbf{E}|X_1| < \infty, \\ 1 & \text{if } \mathbf{E}|X_1| = \infty. \end{cases}$$

(b) Deduce the following converse to the Strong Law of Large Numbers in the case of undefined expectations: If $X_1, X_2, \ldots$ are i.i.d. and $\mathbf{E}X_1$ is undefined (meaning that $\mathbf{E}X_1^+ = \mathbf{E}X_1^- = \infty$) then

$$\mathbf{P}\left(\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n X_k \text{ does not exist} \right) = 1.$$

3. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. Exp(1) random variables. Prove that

$$\mathbf{P}\left(\limsup_{n\to\infty} \frac{X_n}{\log n} = 1 \right) = 1.$$

4. Let X be a r.v. with finite variance, and define a function $M(t) = \mathbf{E}|X - t|$, the "mean absolute deviation of X from t". The goal of this question is to show that the function $M(t)$, like its easier to understand and better-behaved cousin, $\mathbf{E}(X-t)^2$ (the "moment of inertia" around t, which by the Huygens-Steiner theorem is simply a parabola in t, taking its minimum value of $\mathrm{Var}(X)$ at $t = \mathbf{E}X$), also has some nice propreties.

(a) Prove that $M(t) \ge |t - \mathbf{E}X|$.

(b) Prove that $M(t)$ is a convex function.

(c) Prove that

$$\int_{-\infty}^{\infty} \Big(M(t) - |t - \mathbf{E}X| \Big) dt = \mathrm{Var}(X)$$

(see hints below). Deduce in particular that $M(t) - |t - \mathbf{E}X| \xrightarrow[t\to\pm\infty]{} 0$ (again under the assumption that $\mathrm{Var}(X) < \infty$). If it helps, you may assume that X has a density f_X.

(d) Prove that if t_0 is a (not necessarily unique) minimum point of $M(t)$, then t_0 is a median (that is, a 0.5-quantile) of X.

(e) Optionally, draw (or, at least, imagine) a diagram showing the graphs of the two functions $M(t)$ and $|t - \mathbf{E}X|$ illustrating schematically the facts (a)–(d) above.

Hints: For (c), assume first (without loss of generality - why?) that $\mathbf{E}X = 0$. Divide the integral into two integrals, on the positive real axis and the negative real axis. For each of the two integrals, by decomposing $|X - t|$ into a sum of its positive and negative parts and using the fact that $\mathbf{E}X = 0$ in a clever way, show that one may replace the integrand $(\mathbf{E}|X - t| - |t|)$ by a constant multiple of either $\mathbf{E}(X - t)^+$ or $\mathbf{E}(X - t)^-$, and proceed from there.

For (d), first, develop your intuition by plotting the function $M(t)$ in a couple of cases, for example when $X \sim \mathrm{Binom}(1, 1/2)$ and when $X \sim \mathrm{Binom}(2, 1/2)$. Second, if $t_0 < t_1$, plot the graph of the function $x \to \frac{|x-t_1|-|x-t_0|}{t_1-t_0}$, and deduce from this a formula for $M'(t_0+)$ and (by considering $t_1 < t_0$ instead) a similar formula for $M'(t_0-)$, the right- and left-sided derivatives of M at t_0, respectively. On the other hand, think how the condition that t_0 is a minimum point of $M(t)$ can be expressed in terms of these one-sided derivatives.

5. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. (independent and identically distributed) random variables with distribution $U(0, 1)$. Define events $A_1, A_2, \ldots$ by

$$A_n = \{X_n = \max(X_1, X_2, \ldots, X_n)\}$$

(if A_n occurred, we say that n is a **record time**).

(a) Prove that $A_1, A_2, \ldots$ are independent events. Hint: For each $n \geq 1$, let π_n be the random permutation of $(1, 2, \ldots, n)$ obtained by forgetting the values of $(X_1, \ldots, X_n)$ and only retaining their respective order. In other words, define

$$\pi_n(k) = \#\{1 \leq j \leq n : X_j \leq X_k\}.$$

By considering the joint density $f_{X_1,\ldots,X_n}$ (a uniform density on the n-dimensional unit cube), show that π_n is a uniformly random permutation of n elements, i.e. $\mathbf{P}(\pi_n = \sigma) = 1/n!$ for any permutation $\sigma \in S_n$. Deduce that the event $A_n = \{\pi_n(n) = n\}$ is independent of π_{n-1} and therefore is independent of the previous events $(A_1, \ldots, A_{n-1})$, which are all determined by π_{n-1}.

(b) Define

$$R_n = \sum_{k=1}^{n} 1_{A_k} = \#\{1 \leq k \leq n : k \text{ is a record time}\}, \qquad (n = 1, 2, \ldots).$$

Compute $\mathbf{E}(R_n)$ and $\mathrm{Var}(R_n)$. Deduce that for any $\epsilon > 0$ we have that

$$\mathbf{P}\left(\left|\frac{R_n - \log n}{\log n}\right| > \epsilon\right) \xrightarrow[n\to\infty]{} 0.$$

6. Define a constant C by $C = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{6} + \ldots$. A famous result due to Euler (whose proof you may have seen in a calculus or Fourier analysis class) is that $C = \pi^2/6$. Assuming this fact, if X is a random variable that takes positive integer values such that

$$\mathbf{P}(X = n) = \frac{6}{\pi^2} \cdot \frac{1}{n^2} \qquad (n = 1, 2, \ldots),$$

we say that X is distributed according to the **zeta distribution**, and denote $X \sim \text{Zeta}$. Given such a random variable, define random variables $Z_2, Z_3, Z_5, Z_7, \ldots$ by

$$Z_p = \text{the maximal power of } p \text{ that divides } X, \qquad (p \text{ a prime number}).$$

For example, if $X = 84$ then $Z_2 = 2$, $Z_3 = 1$, $Z_7 = 1$ and $Z_p = 0$ for any prime $p \neq 2, 3, 7$.

(a) Prove that for any integer $m \geq 1$,

$$\mathbf{P}(X \text{ is divisible by } m) = \frac{1}{m^2}.$$

(b) Prove that the random variables $(Z_p)_{p \text{ prime}}$ are statistically independent. (Note that it is enough to show that $Z_{p_1}, \ldots, Z_{p_k}$ are independent for any finite set of k primes.)

(c) Deduce the identity, also due to Euler,

$$\prod_{p \text{ prime}} \frac{p^2-1}{p^2} = \frac{3}{4} \cdot \frac{8}{9} \cdot \frac{24}{25} \cdot \frac{48}{49} \cdot \frac{120}{121} \cdot \ldots = \frac{6}{\pi^2} = 0.607927...$$

(d) For each $n \geq 1$, let U_n, V_n be two independent uniformly random numbers in $\{1, \ldots, n\}$. Define a sequence of probabilities

$$s_n = \mathbf{P}(U_n, V_n \text{ are relatively prime}).$$

(Two integers are called relatively prime if they do not have a common divisor; for example 15 and 28 are relatively prime, but 27 and 48 are not relatively prime since they are both divisible by 3.) A theorem from number theory says that $s_n \to 6/\pi^2$ as $n \to \infty$. Explain why this is intuitively to be expected from the above infinite product formula (and for bonus points, prove it, if you can!).

Note. Because of the above result, the constant $6/\pi^2 \approx \%60.8$ is often described loosely as "the probability that two random integers are relatively prime." It can also be described geometrically as "the density of points of the two-dimensional integer lattice that are visible from the origin" — can you see why?