Homework Set No. 6 – Probability Theory (235A), Fall 2013

Due: 11/12/13 in class

1. Let X and $(X_n)_{n=1}^\infty$ be random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Show that if $X_n \xrightarrow[n\to\infty]{\mathbf{P}} X$ (convergence in probability) then there is a subsequence $(X_{n_k})_{k=1}^\infty$ such that $X_{n_k} \xrightarrow[k\to\infty]{\text{a.s.}} X$ (almost sure convergence).

2. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Prove that

$$\int_0^1 \int_0^1 \ldots \int_0^1 f\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right) dx_1\, dx_2 \ldots dx_n \xrightarrow[n\to\infty]{} f(1/2).$$

Hint: Interpret the left-hand side as an expected value; use the laws of large numbers.

3. A bowl contains n spaghetti noodles arranged in a chaotic fashion. Bob performs the following experiment: he picks two random ends of noodles from the bowl (chosen uniformly from the $2n$ possible ends), ties them together, and places them back in the bowl. Then he picks at random two more ends (from the remaining $2n-2$), ties them together and puts them back, and so on until no more loose ends are left.

Let L_n denote the number of **spaghetti loops** at the end of this process (a loop is a chain of one or more spaghettis whose ends are tied to each other to form a cycle). Compute $\mathbf{E}(L_n)$ and $\mathrm{Var}(L_n)$. Find a sequence of numbers $(b_n)_{n=1}^\infty$ such that

$$\frac{L_n}{b_n} \xrightarrow[n\to\infty]{\mathbf{P}} 1,$$

if such a sequence exists.

4. Prove the following one-sided version of Chebyshev's inequality: For any r.v. X and $t \geq 0$,

$$\mathbf{P}(X - \mathbf{E}X \geq t) \leq \frac{\sigma^2(X)}{t^2 + \sigma^2(X)}.$$

Hint: Assume without loss of generality that $\mathbf{E}X = 0$. For any $a > 0$, we have that $\mathbf{P}(X \geq t) \leq \mathbf{P}((X+a)^2 \geq (a+t)^2)$. Bound this using known methods and then look for the value of a that gives the best bound.

5. Let $M_n = (X_{i,j})_{i,j=1}^n$ be a random $n \times n$ matrix of i.i.d. random signs (i.e., random variables such that $\mathbf{P}(X_{i,j} = -1) = \mathbf{P}(X_{i,j} = 1) = 1/2$). Define the random variable $Z_n = \det(M_n)$ (the determinant of M_n). Compute $\mathbf{E}(Z_n)$ and $\mathrm{Var}(Z_n)$.