Homework Set No. 8 – Probability Theory (235A), Fall 2013

Due: 11/25/13 at discussion section

1. Prove that if F and $(F_n)_{n=1}^{\infty}$ are distribution functions, F is continuous, and $F_n(t) \to F(t)$ as $n \to \infty$ for any $t \in \mathbb{R}$, then the convergence is uniform in t.

2. Let $\varphi(x) = (2\pi)^{-1/2} e^{-x^2/2}$ be the standard normal density function.

(a) If $X_1, X_2, \ldots$ are i.i.d. Poisson(1) random variables and $S_n = \sum_{k=1}^n X_k$ (so $S_n \sim$ Poisson(n)), show that if n is large and k is an integer such that $k \approx n + x\sqrt{n}$ then

$$\mathbf{P}(S_n = k) \approx \frac{1}{\sqrt{n}} \varphi(x).$$

Hint: Use the fact that $\log(1+u) = u - u^2/2 + O(u^3)$ as $u \to 0$.

(b) Find $\lim_{n\to\infty} e^{-n} \sum_{k=0}^n \frac{n^k}{k!}$.

(c) If $X_1, X_2, \ldots$ are i.i.d. Exp(1) random variables and denote $S_n = \sum_{k=1}^n X_k$ (so $S_n \sim$ Gamma($n, 1$)), $\hat{S}_n = (S_n - n)/\sqrt{n}$. Show that if n is large and $x \in \mathbb{R}$ is fixed then the density of $\hat{S}_n$ satisfies

$$f_{\hat{S}_n}(x) \approx \varphi(x).$$

3. (a) Prove that if $X, (X_n)_{n=1}^{\infty}$ are random variables such that $X_n \to X$ in probability then $X_n \implies X$.

(b) Prove that if $X_n \implies c$ where $c \in \mathbb{R}$ is a constant, then $X_n \to c$ in probability.

(c) Prove that if $Z, (X_n)_{n=1}^{\infty}, (Y_n)_{n=1}^{\infty}$ are random variables such that $X_n \implies Z$ and $X_n - Y_n \to 0$ in probability, then $Y_n \implies Z$.

4. (a) Let $X_1, X_2, \ldots$ be a sequence of independent r.v.'s that are uniformly distributed on $\{1, \ldots, n\}$. Define

$$T_n = \min\{k : X_k = X_m \text{ for some } m < k\}.$$

If the X_j's represent the birthdays of some sequence of people on a planet in which the calendar year has n days, then T_n represents the number of people in the list who have to declare their birthdays before two people are found to have the same birthday. Show that

$$\mathbf{P}(T_n > k) = \prod_{m=1}^{k-1} \left(1 - \frac{m}{n}\right), \qquad (k \geq 2),$$

and use this to prove that

$$\frac{T_n}{\sqrt{n}} \implies F_{\text{birthday}},$$

where F_{birthday} is the distribution function defined by

$$F_{\text{birthday}}(x) = \begin{cases} 0 & x < 0, \\ 1 - e^{-x^2/2} & x \geq 0 \end{cases}$$

(note: this is not the same as the normal distribution!)

(b) Take $n = 365$. Assuming that the approximation $F_{T_n/\sqrt{n}} \approx F_{\text{birthday}}$ is good for such a value of n, estimate what is the minimal number of students that have to be put into a classroom so that the probability that two of them have the same birthday exceeds 50%. (Ignore leap years, and assume for simplicity that birthdays are distributed uniformly throughout the year; in practice this is not entirely true.)

5. Consider the following two-step experiment: First, we choose a uniform random variable $U \sim U(0,1)$. Then, conditioned on the event $U = u$, we perform a sequence of n coin tosses with bias u, i.e., we have a sequence $X_1, X_2, \ldots, X_n$ such that conditioned on the event $U = u$, the X_k's are independent with the Bernoulli distribution $\text{Ber}(u)$. (Note: the X_k's are independent conditionally on $U = u$, but without the conditioning they are not independent.)

Let $S_n = \sum_{k=1}^n X_k$. Assume that we know that $S_n = k$, but don't know the value of U. What is our subjective estimate of the probability distribution of U given this information? Show that the conditional distribution of U given that $S_n = k$ is the beta distribution $\text{Beta}(k+1, n-k+1)$. In other words, show that

$$\mathbf{P}(U \leq x \mid S_n = k) = \frac{1}{B(k, n-k)} \int_0^x u^k (1-u)^{n-k}\, du, \qquad (0 \leq x \leq 1).$$

Note: This problem has been whimsically suggested by Laplace in the 18th century as a way to estimate the probability that the sun will rise tomorrow, given the knowledge that it has risen in the last n days. See the link: `http://en.wikipedia.org/wiki/Rule_of_succession`.

Hint: Use the following density version of the total probability formula: If A is an event and X is a random variable with density f_X, then

$$\mathbf{P}(A) = \int_{\mathbb{R}} f_X(u)\mathbf{P}(A \mid X = u)\, du.$$

Note that we have not defined what it means to condition on a 0-probability event (this is a somewhat delicate subject that we will not discuss in this course) — but don't worry about it, it is possible to use the formula in computations anyway and get results.