Homework Set No. 9 – Probability Theory (235A), Fall 2013

Due: 12/2/13 at discussion section

Happy Thanksgiving!

1. Compute the characteristic functions for the following distributions.

(a) **Poisson distribution:** $X \sim \text{Poisson}(\lambda)$.

(b) **Geometric distribution:** $X \sim \text{Geom}(p)$.

(c) **Uniform distribution:** $X \sim U[a, b]$, and in particular $X \sim [-1, 1]$ which is especially symmetric and useful in applications.

(d) **Exponential distribution:** $X \sim \text{Exp}(\lambda)$.

(e) **Symmetrized exponential:** A r.v. Z with density function $f_Z(x) = \frac{1}{2}e^{-|x|}$. (This is the distribution of the r.v. $X - Y$ when $X, Y \sim \text{Exp}(1)$ are independent. It is also the distribution of $\varepsilon \cdot X$ where $X \sim \text{Exp}(1)$ and $\varepsilon = \pm 1$ is an unbiased random sign independent of X.)

(f) **Cauchy distribution:** the distribution of an r.v. with density function $f_X(x) = \frac{1}{\pi}\frac{1}{1+x^2}$. (This distribution appeared for example in problem 4 of HW 2.)

Hint. Use the computation of part (e) above together with the Fourier inversion formula (Theorem 13.6 on page 87 in the lecture notes).

2. Using the result of problem 1(f) above, prove that if X, Y are independent r.v.s with the Cauchy distribution then for any $\alpha \in [0, 1]$, $\alpha X + (1 - \alpha)Y$ is also a Cauchy r.v.

(In particular, the average $(X + Y)/2$ of two independent Cauchy r.v.s is a Cauchy r.v., and more generally the empirical average $(X_1 + \ldots + X_n)/n$ of n i.i.d. Cauchy r.v.s is again a Cauchy r.v.)

3. (a) Let $Z_1, Z_2, \dots$ be a sequence of independent r.v.'s such that the random series $X = \sum_{n=1}^{\infty} Z_n$ converges a.s. Prove that

$$\varphi_X(t) = \prod_{n=1}^{\infty} \varphi_{Z_n}(t), \qquad (t \in \mathbb{R}).$$

(b) Apply this result to the binary expansion $X = \sum_{n=1}^{\infty} \frac{Y_n}{2^n}$ of a random variable $X \sim U(0,1)$, in which the binary digits $Y_1, Y_2, \dots \in \{0,1\}$ are i.i.d. Ber(1/2) random variables, to prove the identity

$$\frac{\sin(t)}{t} = \prod_{n=1}^{\infty} \cos\left(\frac{t}{2^n}\right), \qquad (t \in \mathbb{R}).$$

(c) Substitute $t = \pi/2$ in the identity above to prove the formula

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots$$