1. Prove by induction that for every $n \in \mathbb{N}$

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Proof. When $n = 1$, the statement becomes

$$1 = 1^2 = \frac{1(1+1)(2+1)}{6} = \frac{6}{6} = 1,$$

which is clearly true.

Now, assume that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

We want to show that

$$1^2 + 2^2 + \cdots + n^2 + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}.$$

Now,

$$\begin{aligned} 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 &= \left(1^2 + 2^2 + \cdots + n^2\right) + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + \frac{6n^2 + 12n + 6}{6} = \frac{n(n+1)(2n+1) + 6n^2 + 12n + 6}{6} \\ &= \frac{2n^3 + 3n^2 + n + 6n^2 + 12n + 6}{6} = \frac{2n^3 + 7n^2 + 6n + 2n^2 + 7n + 6}{6} \\ &= \frac{n(2n^2 + 7n + 6) + (2n^2 + 7n + 6)}{6} = \frac{(n+1)(2n^2 + 7n + 6)}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6}, \end{aligned}$$

as desired. □

2. Let $x \in \mathbb{R}$ and $x > 0$. Prove by induction that for all $n \in \mathbb{N}$

$$(1 + x)^n \geq 1 + nx.$$

Proof. When $n = 1$, the statement is simply $1 + x \geq 1 + x$, which is clearly true. So, assume

$$(1 + x)^n \geq 1 + nx.$$

Then, we want to show

$$(1 + x)^{n+1} \geq 1 + (n+1)x.$$

Now,

$$(1+x)^{n+1} = (1+x)(1+x)^n \geq (1+x)(1+nx) = 1 + nx + x + nx^2 = 1 + (n+1)x + nx^2 \geq 1 + (n+1)x,$$

where in the first inequality in this chain of inequalities we used the inductive hypothesis together with the fact that $1 + x > 0$ (since x was assumed to be a positive number), and in the last inequality we used the fact that $nx^2 \geq 0$. □

3. Prove by induction that for all $n \in \mathbb{N}$, the number $7^n - 4^n$ is divisible by 3.

Proof. The case when $n = 1$ is clear. So, assume $7^n - 4^n$ is divisible by 3. Then,

$$7^{n+1} - 4^{n+1} = 7 \cdot 7^n - 4 \cdot 4^n = (3 + 4) \cdot 7^n - 4 \cdot 4^n = 3 \cdot 7^n + 4 \cdot (7^n - 4^n),$$

which is divisible by 3 since the first term is a multiple of 3, and we assumed $7^n - 4^n$ was divisible by 3. □

4. Find and explain the flaw in the proof by induction that was presented in class of the claim that all giraffes are of the same height.

   Recall that induction requires that if a statement is true for the nth case, it must also be true for the $n+1$th case for all values of n. The proof that was given in class breaks down in the case $n = 1$, since in this case $n + 1 = 2$ and there is no "giraffe #3" to use as an intermediary that shows that "giraffe #1" and "giraffe #2" are the same height. If, however, the statement were true for $n = 2$, it would be true for any value of n.

   Note that with proofs by induction, it is not enough that the proof of the inductive step "if $P(n)$ is true then $P(n+1)$ is true" holds for *most* values of n; if the proof breaks down even for a single value of n (as happens here with $n = 1$), this breaks the chain of inductive reasoning and the entire result can become catastrophically false.

5. Express the following statements in words, determine whether or not they are true, and then form their negations.

   (a) $\forall x \in \mathbb{R}, x \geq 0$.

   Translation: all real numbers are non-negative.

   The statement is false.

   Negation: there exists a negative real number.

   (b) $\exists x \in \mathbb{R}, x \geq 0$.

   Translation: there is a non-negative real number.

   This statement is true.

   Negation: all reals numbers are negative.

   (c) $\forall x \in \mathbb{R}, x^2 \geq 0$.

   Translation: the square of a real number is always non-negative.

   This statement is true.

   Negation: there is a real number with the property that its square is negative.

   (d) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y = 1$.

   Translation: the sum of any two real numbers is equal to 1.

   This statement is false.

   Negation: there are two real numbers whose sum is not equal to 1.

   (e) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 1$.

   Translation: For any real number, there is another real number with the property that the sum of these two numbers is 1.

   This statement is true.

   Negation: There is a real number with the property that its sum with any other real number is not equal to 1.

   (f) $\exists x \in \mathbb{X}, \forall y \in \mathbb{R}, x + y = 1$.

   Translation: there is a real number whose sum with every real number is always equal to 1.

   This statement is false.

   Negation: for any real number there is a real number such that the sum of the two is not equal to 1.

   (g) $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 1$.

   Translation: There are a pair of real numbers whose sum is 1.

   This statement is true.

   Negation: There is no pair of real numbers which sum to 1.

6. Show that $\mathbb{N} \times \mathbb{N}$ is countable.

Proof. Consider the following arrangement of the elements of $\mathbb{N} \times \mathbb{N}$:

$$
\begin{array}{|cccccc}
\hline
(1,1) & (1,2) & (1,3) & \cdots & (1,n) & \cdots \\
(2,1) & (2,2) & (2,3) & \cdots & (2,n) & \cdots \\
(3,1) & (3,2) & (3,3) & \cdots & (3,n) & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\
(n,1) & (n,2) & (n,3) & \cdots & (n,n) & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots
\end{array}
$$

One can list all the elements in an infinite list by scanning this table along diagonals that go from north-east to south-west: the first diagonal contains just $(1,1)$; the second diagonal contains $(1,2)$ and $(2,1)$; the third diagonal contains $(1,3),(2,2)$, and $(3,1)$, etc. This produces the list

$$(1,1),(1,2),(2,1),(1,3),(2,2),(3,1),(1,4),(2,3),(3,2),\ldots$$

It is obvious that this list covers every pair (m,n) of natural numbers in the table. We have shown how to enumerate all the elements of $\mathbb{N} \times \mathbb{N}$ in an infinite list, and therefore it is a countable set. $\square$