Math 25 — Homework Assignment #3

Homework due: **Tuesday 4/19/11** at beginning of discussion section

Reading material. Read sections 1.1–1.6 in the textbook.

Problems

1. Let F be a field (a set with operations $+$ and $\cdot$ satisfying the field axioms). Prove the following statements about F. Use only the field axioms and facts proved in class, rather than any intuitive assumptions about properties of addition and multiplication that you are familiar with. At each step of the proof state exactly which axiom or known fact you are using.
   (a) For any $a, b, c \in F$, $c(a+b) = ca + cb$.
   (b) For any $a, b \in F$, if $ab = 0$ then either $a = 0$ or $b = 0$.
   (c) For any $a, b, c \in F$, if $ac = bc$ and $c \neq 0$ then $a = b$.

2. ("Clock arithmetic") Let $\mathbb{Z}_{12}$ denote the set $\{0, 1, 2, \ldots, 11\}$, with operations of addition and multiplication defined by
$$x \oplus y = \text{ the remainder left by } x + y \text{ on division by 12,}$$
$$x \odot y = \text{ the remainder left by } x \cdot y \text{ on division by 12.}$$
   Show that $\mathbb{Z}_{12}$ is not a field. Note that this can be done by showing that $\mathbb{Z}_{12}$ does not satisfy at least one of the field axioms, or alternatively by showing that $\mathbb{Z}_{12}$ does not satisfy one of the properties that we proved (in class or in Problem 1 above) that a field must satisfy.

3. Let x, y be positive real numbers. Prove the arithmetic-geometric mean inequality
$$\sqrt{xy} \leq \frac{x+y}{2}.$$
   Use only the order axioms of $\mathbb{R}$ (Section 1.4) and standard algebra.[1]

4. Answer problem 1.6.2 in the textbook.

5. Let $E \subset \mathbb{R}$ be a set for which $\max E$ exists. Prove that $\max E = \sup E$, or show a counterexample.

[1] For a hint, see note 3 at the end of Chapter 1 in the textbook.