Math 25 — Solutions to Homework Assignment #3

Solutions

1. Let F be a field.

   (a) Fix some $a, b, c \in F$. By commutativity of multiplication (axiom A1), distributivity of multiplication over addition (axiom AM1), and commutativity of multiplication again (A1), respectively, $c(a+b) = (a+b)c = ac+bc = ca+cb$. $\square$

   (b) Fix some $a, b \in F$ such that $ab = 0$ and assume (without loss of generality) $a \neq 0$. Then by the existence of a multiplicative inverse (axiom M4) a has an inverse element a^{-1} such that $a^{-1}a = 1$. By a claim proved in class, $x \cdot 0 = 0$ for any $x \in F$. Applying this with $x = a^{-1}$ and then using associativity (M2) and the defining property (M3) of 1 gives that $0 = a^{-1} \cdot 0 = a^{-1}(ab) = (a^{-1}a)b = 1 \cdot b = b$. Thus, $b = 0$. $\square$

   (c) Fix some $a, b, c \in F$, with $ac = bc$ and $c \neq 0$. By the existence of multiplicative inverse, substitution, and assicativity of multiplication, $ac = bc \Longrightarrow (ac)c^{-1} = (bc)c^{-1} \Longrightarrow a(cc^{-1}) = b(cc^{-1}) \Longrightarrow a1 = b1 \Longrightarrow a = b$. $\square$

2. If $\mathbb{Z}_{12}$ is a field, then as shown in problem 1. (b), for any $a, b \in \mathbb{Z}_{12}$, $ab = 0$ implies that either $a = 0$ or $b = 0$. However, $3 \cdot 4 = 12$, so that in $\mathbb{Z}_{12}$, $3 \odot 4 = 0$, while $3 \neq 0$ and $4 \neq 0$. Thus, $\mathbb{Z}_{12}$ cannot be a field. $\square$

3. We begin by proving two useful results:

   Result 1
   For any $x \in \mathbb{R}$, $x^2 \geq 0$.

   Proof. This is trivial if $x = 0$. Instead, let $x > 0$. Then:

   $$x \cdot x > 0 \cdot x = 0 \quad \text{(O4)}$$

   On the other hand, let $x < 0$. In this case,

   $$0 = x + (-x) < 0 + (-x) = -x \quad \text{(O3)}$$

   So that, using the result for $x > 0$, $x^2 = (-x)^2 > 0$. $\square$

Result 2
For any positive $x, y \in \mathbb{R}$ with $x > y$, $x^2 > y^2$.

Proof. Use axiom O4 twice:

$$x^2 = x \cdot x > y \cdot x = x \cdot y > y \cdot y = y^2.$$

$\square$

Now, consider any positive $x, y \in \mathbb{R}$. Then $x - y \in \mathbb{R}$, so $(x-y)^2 \geq 0$, as shown above. But, if $(x-y)^2 > 0$:

$$\begin{aligned} 0 < (x-y)^2 &= x^2 - 2xy + y^2 \\ 4xy &< x^2 + 2xy + y^2 && \text{(O3)} \\ xy &< \frac{(x+y)^2}{4} && \text{(O4)} \\ \sqrt{xy} &< \frac{x+y}{2} && \text{(contrapos. of Result 2)} \end{aligned}$$

If $(x-y)^2 = 0$, then $x = y$ in which case $\sqrt{xy} = x = \frac{x+y}{2}$. $\square$

4. (a) $\sup E = \infty$; $\inf E = 1 = \min E$; $\max E$ does not exist.
   (b) $\sup E = \infty$; $\inf E = -\infty$; $\max E, \min E$ do not exist.
   (c) $\sup E = \infty$; $\inf E = -\infty$; $\max E, \min E$ do not exist.
   (d) $\sup E = \infty$; $\inf E = -\infty$; $\max E, \min E$ do not exist.
   (e) $\sup E = 7 = \max E$; $\inf E = -3 = \min E$
   (f) $\sup E = \sqrt{2}$; $\inf E = -\sqrt{2}$; $\max E, \min E$ do not exist.
   (g) $\sup E = \frac{1+\sqrt{5}}{2}$; $\inf E = \frac{1-\sqrt{5}}{2}$; $\max E, \min E$ do not exist.
   (h) $\sup E = 1 = \max E$; $\inf E = 0$; $\min E$ does not exist.
   (i) $\sup E = \sqrt[3]{3} = \max E$; $\inf E = 1 = \min E$.

5. $\sup E = s$ and, because we assume it to exist, set $\max E = m$. m is then an upper bound to E. Because, by definition, the supremum is bounded above by any other upper bound on E, $s \leq m$. Moreover, since $m \in E$, and s is itself an upper bound to E, $m \leq s$. Thus, $m = s$.