# Math 25 — Homework Assignment #4

**Homework due: Tuesday 4/26/11** at beginning of discussion section

**Reading material.** Read sections 1.7, 1.9, 1.10, 2.2, 2.4 in the textbook.

**Problems**

1. Using the Archimedean Theorem, prove each of the three statements that follow the proof of that theorem in section 1.7 of the textbook.

2. The completeness axiom can be used to construct numbers whose existence we only suspected before. As an illustration of this idea, prove that the number
$$\alpha = \sup\{x \in \mathbb{R} : x^2 < 2\}$$
exists in $\mathbb{R}$ and satisfies $\alpha^2 = 2$. (Hint: show that either of the assumptions $\alpha^2 < 2$ and $\alpha^2 > 2$ lead to a contradiction.)

3. A set $E$ of real numbers is called *ultra-dense* if for every real numbers $a < b$, the interval $(a, b)$ contains infinitely many elements of $E$. Prove that $E$ is ultra-dense if and only if it is dense.

4. For real numbers $x, y$, show that $\max\{x, y\} = (x + y)/2 + |x - y|/2$. What analogous expression involving absolute values would give the minimum $\min\{x, y\}$? (Hint: think geometrically.)

5. For a real number $x$, define quantities $x_+$ (the "positive part" of $x$) and $x_-$ (the "negative part" of $x$) by
$$x_+ = \max(x, 0) = \begin{cases} x & x \geq 0, \\ 0 & x < 0, \end{cases} \quad x_- = \max(-x, 0) = \begin{cases} 0 & x \geq 0, \\ -x & x < 0. \end{cases}$$
Prove the following identities:
$$\begin{array}{llll} \text{(a)} & x = x_+ - x_- & \text{(c)} & x_+ = \frac{1}{2}(|x| + x) \\ \text{(b)} & |x| = x_+ + x_- & \text{(d)} & x_- = \frac{1}{2}(|x| - x) \end{array}$$

6. Consider the sequence $\{F_n\}_{n \geq 1}$ defined by the following properties: (i) the first two terms are $F_1 = 1$, $F_2 = 1$, and (ii) for each $n \geq 2$, $F_n$ is defined as the sum of the two previous terms, i.e., $F_n = F_{n-1} + F_{n-2}$.

   (a) Compute the first 10 terms of this sequence.

(b) Use induction to show that $F_n = G_n$ for all $n \geq 1$, where $G_n$ is defined by

$$G_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{1-\sqrt{5}}{2} \right)^n \right].$$

Hint: first, check that $F_1 = G_1$ and $F_2 = G_2$; this will give the induction base. Next, show that if for some $n \geq 2$ we assume that $F_{n-1} = G_{n-1}$ and $F_{n-2} = G_{n-2}$, then it would follow that $F_n = G_n$. Formulate carefully your inductive hypothesis and explain why this step implies the result using the induction principle. For the inductive step, it is recommended to make use of the fact that both of the numbers $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$ are solutions of the quadratic equation $x^2 = x + 1$.

7. Give a precise $\varepsilon, N$ argument to prove the existence of $\lim_{n\to\infty} \frac{2n+3}{3n+4}$. That is, you must identify the limit $L$, and then prove that the statement "$\frac{2n+3}{3n+4}$ converges to $L$ as $n \to \infty$" holds.