Math 25 — Solutions to Homework Assignment #4

1. Using the Archimedean Theorem, prove each of the three statements that follow the proof of that theorem in section 1.7 of the textbook.

   (a) No matter how large a real number x is given, there is always a natural number n larger.

   Proof. Suppose that there is some x such that no natural number is larger than x. Then x is an upper bound for the set of natural numbers, which contradicts the Archimedean Property. □

   (b) Given any positive number y, no matter how large, and any positive number x, no matter how small, there is some natural number n such that $nx > y$.

   Proof. Assume there is no such n. That is $nx \leq y$ for all n. Note that this is equivalent to $n \leq yx^{-1}$ for all n, which contradicts item (a). □

   (c) Given any positive number x, no matter how small, once can always find a fraction $\frac{1}{n}$ such that $\frac{1}{n} < x$.

   Proof. Assume there is no such n. Then $\frac{1}{n} \geq x$ for all n. Note that this is equivalent to $n \leq x^{-1}$ for all n, which contradicts item (a). □

2. The completeness axiom can be used to construct numbers whose existence we only suspected before. As an illustration of this idea, prove that the number

$$\alpha = \sup\{x \in \mathbb{R} : x^2 < 2\}$$

exists in $\mathbb{R}$ and satisfies $\alpha^2 = 2$. (Hint: show that either of the assumptions $\alpha^2 < 2$ and $\alpha^2 > 2$ lead to a contradiction.)

Proof. We will use the following auxiliary result: if $x^2 < y^2$, then $x < y$ where $x, y > 0$. To see this, note that $x^2 < y^2$ is equivalent to $x^2 - y^2 < 0$ or $(x - y)(x + y) < 0$. Since $x, y > 0$, we have that $x + y \neq 0$, so we may multiply this inequality by $(x + y)^{-1}$, yielding $x - y < 0$. Note that this is equivalent to $x < y$.

Now, let $A = \{x \in \mathbb{R} : x^2 < 2\}$ and $\alpha = \sup A$. Suppose $\alpha^2 < 2$. Then $\alpha < \sqrt{2}$. Note that $\alpha < \frac{\alpha+\sqrt{2}}{2} < \sqrt{2}$, which contradicts the fact that α is an upper bound of A. Suppose $\alpha^2 > 2$. Then $\alpha > \frac{\alpha+\sqrt{2}}{2} > \sqrt{2}$, which contradicts the fact that α is the least upper bound of A. □

3. A set E of real numbers is called *ultra-dense* if for every real numbers $a < b$, the interval (a, b) contains infinitely many elements of E. Prove that E is ultra-dense if and only if it is dense.

Proof. Clearly, if E is ultra-dense, then it is also dense. So suppose E is dense in $\mathbb{R}$. Assume there is some interval (a, b) such that there are only finitely many elements of E in (a, b). Denote these elements $\{e_1, \ldots, e_n\}$. Without loss of generality, it may be assumed that $a < e_1 < e_2 < \cdots < e_n < b$. Consider the interval (a, e_1). Since E is dense, there is some $e_0 \in E \cap (a, e_1)$. But this implies $e_0 \in (a, b)$—a contradiction. So there must be infinitely many elements of E in any open interval. □

4. For real numbers x, y, show that $\max\{x, y\} = (x + y)/2 + |x - y|/2$. What analogous expression involving absolute values would give the minimum $\min\{x, y\}$? (Hint: think geometrically.)

Proof. Suppose $x = y$. Then the expression above becomes

$$\max\{x, x\} = \frac{x + x}{2} + \frac{|x - x|}{2} = \frac{2x}{2} = x,$$

which is clearly true. Suppose $x \neq y$. Without loss of generality, it may be assumed that $y < x$. Then

$$\frac{x + y}{2} + \frac{|x - y|}{2} = \frac{x + y}{2} + \frac{x - y}{2} = \frac{2x}{2} = x = \max\{x, y\}.$$

Recall that if $x < y$, then $-x > -y$. This implies that $\max\{x, y\} = -\min\{-x, -y\}$ or $\min\{x, y\} = -\max\{-x, -y\}$. Using the formula above, we obtain

$$\min\{x, y\} = -\max\{-x, -y\} = -\frac{-x - y}{2} - \frac{|-x + y|}{2} = \frac{x + y}{2} - \frac{|x - y|}{2}.$$

□

5. For a real number x, define quantities x_+ (the "positive part" of x) and x_- (the "negative part" of x) by

$$x_+ = \max(x, 0) = \begin{cases} x & x \geq 0, \\ 0 & x < 0, \end{cases} \quad x_- = \max(-x, 0) = \begin{cases} 0 & x \geq 0, \\ -x & x < 0. \end{cases}$$

Prove the following identities:

$$\begin{array}{llll} \text{(a)} & x = x_+ - x_- & \text{(c)} & x_+ = \frac{1}{2}(|x| + x) \\ \text{(b)} & |x| = x_+ + x_- & \text{(d)} & x_- = \frac{1}{2}(|x| - x) \end{array}$$

Proof. First, note that when $x = 0$, we have $x_+ = 0 = x_-$, and the identities are obvious. Suppose $x > 0$. Then $x_+ = x$, $x_- = 0$, and $|x| = x$, and the identities follow easily. Similarly, if $x < 0$ we have $x_+ = 0, x_- = -x$, and $|x| = -x$, and the identities follow easily. □

6. Consider the sequence $\{F_n\}_{n\geq 1}$ defined by the following properties: (i) the first two terms are $F_1 = 1$, $F_2 = 1$, and (ii) for each $n \geq 2$, F_n is defined as the sum of the two previous terms, i.e., $F_n = F_{n-1} + F_{n-2}$.

   (a) Compute the first 10 terms of this sequence.

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

   (b) Use induction to show that $F_n = G_n$ for all $n \geq 1$, where G_n is defined by

$$G_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

Proof. First, we show that this result holds for the cases $n = 1$ and $n = 2$. When $n = 1$ we have

$$G_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right] = \frac{1}{\sqrt{5}}\frac{2\sqrt{5}}{2} = 1.$$

When $n = 2$ we obtain

$$G_2 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right] = \frac{1}{\sqrt{5}}\left[\left(\frac{1+2\sqrt{5}+5-1+2\sqrt{5}-5}{4}\right)\right] = \frac{1}{\sqrt{5}}\frac{4\sqrt{5}}{4} = 1.$$

Now, suppose that

$$F_{n-1} = G_{n-1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right]$$

and

$$F_n = G_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

Then,

$$\begin{aligned}
F_{n+1} = F_n + F_{n-1} &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(1 + \frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(1 + \frac{1-\sqrt{5}}{2}\right)\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{3-\sqrt{5}}{2}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{6+2\sqrt{5}}{4}\right)-\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{6-2\sqrt{5}}{4}\right)\right]\\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{1+\sqrt{5}}{2}\right)^{2}-\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{1-\sqrt{5}}{2}\right)^{2}\right]\\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right],
\end{aligned}
$$

as desired. □

Note. In the proof above, the inductive hypothesis is that the proposed formula for F_n works for two successive values $n-1$ and n. From this, we show that the formula holds also for $n+1$. That, combined with the knowledge that the formula holds for n, gives us two new successive values n and $n+1$ for which we know the formula holds, so it is enough to complete the induction step. Note also that the base of the induction requires checking the formula for the first two values, $n=1$ and $n=2$.

7. Give a precise ε, N argument to prove the existence of $\lim_{n\to\infty}\frac{2n+3}{3n+4}$. That is, you must identify the limit L, and then prove that the statement "$\frac{2n+3}{3n+4}$ converges to L as $n\to\infty$" holds.

Proof. Our intuition from calculus leads us to believe that $L=\frac{2}{3}$, but we must prove this. Now, let $\varepsilon>0$ be given. Then, by the Archimedean Property, there is some natural number N such that $N>\frac{\frac{1}{\varepsilon}-12}{9}$. Let $n\geq N$. Then

$$
\begin{aligned}
n>\frac{\frac{1}{\varepsilon}-12}{9} &\iff 9n>\frac{1}{\varepsilon}-12 \iff 9n+12>\frac{1}{\varepsilon} \iff \frac{1}{9n+12}<\varepsilon \iff \left|\frac{1}{9n+12}\right|<\varepsilon\\
&\iff \left|\frac{6n-6n+1}{9n+12}\right|<\varepsilon \iff \left|\frac{6n-6n+9-8}{9n+12}\right|<\varepsilon \iff \left|\frac{6n+9}{9n+12}-\frac{6n+8}{9n+12}\right|<\varepsilon\\
&\iff \left|\frac{3(2n+3)}{3(3n+4)}-\frac{2(3n+4)}{3(3n+4)}\right|<\varepsilon \iff \left|\frac{2n+3}{3n+4}-\frac{2}{3}\right|<\varepsilon,
\end{aligned}
$$

as desired. □