Math 25 — Homework Assignment #5

Note. **This homework will not be collected or graded.** It is meant as a set of practice problems to aid in studying for the midterm exam. Solutions for selected problems will be posted separately later this week, but you are strongly encouraged to work out the solutions yourself before looking at the published ones; doing so will greatly increase your chances of success in the midterm and final and thereby indirectly help you to improve your grade.

Reading material. Read sections 2.4–2.7 in the textbook.

Problems on previous material

1. Answer the following problems in the textbook: A.7.1, A.7.3, A.8.2, A.8.5, 1.9.2, 1.9.6, 1.10.3, 1.10.5

Problems on the past week's material

2. Prove that the limit $\lim_{n\to\infty} n^2$ does not exist.

3. Decide if each of the following sequences $(a_n)_{n=1}^{\infty}$ converges or diverges. If the sequence converges, state its limit. In either case, you must use the appropriate definition or theorem to prove that the sequence converges to the claimed limit or that the sequence diverges.

   (a) $a_n = \frac{1}{n^2}$

   (b) $a_n = \frac{n^2+n}{n^2}$

   (c) $a_n = \frac{n^2+n}{n}$

   (d) $a_n = \cos(n\pi)$

   (e) $a_n = \frac{(-1)^n}{n}$

4. In each of the following statements, $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ represent sequences of real numbers. For each statement, decide if it is true or false. If it is true, give a proof. If it is false, provide a counterexample.

   (a) If (a_n) and (b_n) are both bounded, then so is $(a_n + b_n)$.

   (b) If (a_n) and (b_n) are both unbounded, then so is $(a_n + b_n)$.

   (c) If (a_n) and (b_n) are both bounded, then so is $(a_n \cdot b_n)$.

(d) If (a_n) is bounded, then a_n is convergent.

(e) If (a_n) and (b_n) are both divergent, then so is $(a_n + b_n)$.

5. Prove that the sequence $b_n = 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n}$ is bounded.

   (**Hint:** Compute the first few terms of the sequence as rational fractions. From the computation, guess a simple formula for b_n. Prove that formula by induction, and use it to prove that the sequence is bounded.)

6. Prove that the sequence $c_n = (-1)^n \sqrt{n}$ is unbounded. Does it have a limit in the generalized sense? (i.e., does it diverge to $+\infty$ or to $-\infty$?)

7. Suppose (a_n) and (b_n) are sequences of real numbers such that (a_n) is bounded and (b_n) diverges to infinity. Prove that

$$\lim_{n \to \infty} \frac{a_n}{b_n} = 0.$$