Solutions to homework assignment #5

Reading material. Read sections 2.4–2.7 in the textbook.

Problems on previous material

1. Answer the following problems in the textbook: A.7.1, A.7.3, A.8.2, A.8.5, 1.9.2, 1.9.6, 1.10.3, 1.10.5

Solution to A.7.1. Let $n = 3$. For this n, the equation $4x^2+x-n = 0$ becomes $4x^2 + x - 3 = 0$. This equation has a rational root $x = -1$.

Solution to A.7.3. The converse: "Every continuous function is differentiable" (this is false — e.g., the function $f(x) = |x|$ is continuous but not differentiable). The contrapositive: "Every function that is not continuous is also not differentiable". This is true.

Solution to A.8.2. I claim that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$. The proof is by induction on n. For $n = 1$, the left-hand side is 1, and $n^2 = 1$, so the statement is true. Assume that it is true for $n-1$. This means that

$$1+3+5+\ldots+\big(2(n-1)-1\big) = 1+3+5+\ldots+(2n-3) = (n-1)^2.$$

It follows, using the induction hypothesis, that

$$\begin{aligned}1 + 3 + 5+ \ldots + (2n - 1) &= (1 + 3 + 5 + \ldots + (2n - 3)) + (2n - 1)\\ &= (n - 1)^2 + (2n - 1) = n^2 - 2n + 1 + (2n - 1) = n^2.\end{aligned}$$

Solution to 1.10.5. For numbers x, y, denote $a = y, b = x - y$. Applying the triangle inequality $|a + b| \le |a| + |b|$ to a and b, we get

$$|x| \le |y| + |x - y|,$$

or $|x - y| \ge |x| - |y|$. Similarly, reversing the roles of x and y (or redefining a and b as $a = x$, $b = y - x$) gives that $|x - y| \ge |y| - |x| = -(|x| - |y|)$. Combining these two facts gives

$$|x - y| \ge \max\Big\{|x| - |y|, |y| - |x|\Big\} = \Big||x| - |y|\Big|.$$

Problems on the past week's material

2. Prove that the limit $\lim_{n\to\infty} n^2$ does not exist.

Solution 1. Let $L \in \mathbb{R}$. We claim that the sequence $s_n = n^2$ does not converge to L. To show this, take $\varepsilon = 1$. Since $s_n = n^2 \geq n$, if $n > |L| + 1$ we have that

$$|s_n - L| \geq |s_n| - |L| \geq n - |L| > 1 = \varepsilon.$$

It follows that for any $N \in \mathbb{N}$, if we take $n > \max\{|L| + 1, N\}$ then $|s_n - L| > \varepsilon$. Thus by the definition of the limit (s_n) does not converge to L.

Solution 2. Show that (n^2) is an unbounded sequence. It follows by a theorem we proved in class that (n^2) is a divergent sequence.

3. Decide if each of the following sequences $(a_n)_{n=1}^{\infty}$ converges or diverges. If the sequence converges, state its limit. In either case, you must use the appropriate definition or theorem to prove that the sequence converges to the claimed limit or that the sequence diverges.

   (a) $a_n = \frac{1}{n^2}$

   Solution. We showed in class that $b_n = 1/n$ is a sequence that converges to 0. Note that $a_n = b_n \cdot b_n$. Therefore, by the theorem we proved on the limit of a product of two convergent sequences, we get that

   $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \cdot \lim_{n\to\infty} b_n = 0 \cdot 0 = 0.$$

   (b) $a_n = \frac{n^2+n}{n^2}$

   Solution. $a_n = 1 + \frac{1}{n}$. We have $\lim_{n\to\infty} 1 = 1, \lim_{n\to\infty} \frac{1}{n} = 0$, so by the theorem on the addition of two convergent sequences, a_n converges and its limit is 1.

   (c) $a_n = \cos(n\pi)$

   Solution. $\cos(n\pi) = (-1)^n$. This sequence alternately takes the values 1 and -1. It does not converge. To prove this, let L be a hypothetical limit. To prove that $a_n \not\to L$ as $n \to \infty$, divide into two cases: $L \geq 0$ or $L < 0$. If $L \geq 0$, note that $|a_n - L| = |(-1) - L| = L + 1 > 1/2$ for odd values of n, and if $L < 0$, then $|a_n - L| = |1 - L| > 1/2$ for even values of n. In either case, this shows the required property that there exists an $\varepsilon > 0$ (in this case $\varepsilon = 1/2$) for which, for all values of $N \in \mathbb{N}$ there is an $n \geq N$ such that $|a_n - L| > \varepsilon$.

(d) $a_n = \frac{(-1)^n}{n}$

4. In each of the following statements, $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ represent sequences of real numbers. For each statement, decide if it is true or false. If it is true, give a proof. If it is false, provide a counterexample.

   (a) If (a_n) and (b_n) are both bounded, then so is $(a_n + b_n)$.

   Solution. This is true, since if $|a_n| \leq M$ and $|b_n| \leq M'$ for all n, then by the triangle inequality,

   $$|a_n + b_n| \leq |a_n| + |b_n| \leq M + M'.$$

   (b) If (a_n) and (b_n) are both unbounded, then so is $(a_n + b_n)$.

   Solution. False. As a counterexample, consider $a_n = n$ and $b_n = -n$.

   (c) If (a_n) and (b_n) are both bounded, then so is $(a_n \cdot b_n)$.

   Solution. True. The proof is a variation on (a) above.

   (d) If (a_n) is bounded, then a_n is convergent.

   Solution. False. $a_n = (-1)^n$ is bounded but (as shown above) is not convergent.

   (e) If (a_n) and (b_n) are both divergent, then so is $(a_n + b_n)$.

   Solution. False, see (b) above for a counterexample.

5. Prove that the sequence $b_n = 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n}$ is bounded.

   Solution sketch. By induction one shows that $b_n = 2 - \frac{1}{2^n}$. It follows that $0 < b_n < 2$ for all n.

6. Prove that the sequence $c_n = (-1)^n \sqrt{n}$ is unbounded. Does it have a limit in the generalized sense? (i.e., does it diverge to $+\infty$ or to $-\infty$?)

7. Suppose (a_n) and (b_n) are sequences of real numbers such that (a_n) is bounded and (b_n) diverges to infinity. Prove that

   $$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

Solution. Let $\varepsilon > 0$. Since (a_n) is bounded, there exists a number M such that $|a_n| \leq M$ for all n. Since (b_n) diverges to ∞, there exists an $N \in \mathbb{N}$ such that for any $n \geq N$,

$$b_n \geq \frac{M}{\varepsilon}.$$

Therefore for any $n \geq N$, we have

$$\left|\frac{a_n}{b_n}\right| = \frac{|a_n|}{|b_n|} \leq \frac{M}{M/\varepsilon} = \varepsilon.$$

This proves that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.