Math 25 – Solutions to Homework Assignment #6

1. Let sequences $(s_n)_{n=1}^\infty$ and $(t_n)_{n=1}^\infty$ converge to respective limits S and T. Fix some $\epsilon > 0$. By definition, there exist numbers m_s, m_t such that for any $n_s \geq m_s$ and $n_t \geq m_t$, $|s_{n_s} - S| < \frac{\epsilon}{2}$ and $|t_{n_t} - T| < \frac{\epsilon}{2}$. If $n \geq \max\{m_s, m_t\}$, by the triangle inequality, $|(s_n - t_n) - (S - T)| = |(s_n - S) - (t_n - T)| \leq |s_n - S| + |t_n - T| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. By definition, then, $s_n - t_n \longrightarrow S - T$ as $n \longrightarrow \infty$.

2. For $(s_n)_{n=1}^\infty$ and $(t_n)_{n=1}^\infty$ such that

$$s_n = \sqrt{n+1}, \quad t_n = \sqrt{n},$$

Both sequences diverge to infinity, and hence the reasoning in problem 1. does not apply. However, for any n, since $a - b = (\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})$,

$$s_n - t_n = \sqrt{n+1} - \sqrt{n} = \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

Now, since

$$0 \leq \frac{1}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{\sqrt{n}}$$

And because $\frac{1}{\sqrt{n}} \longrightarrow 0$ as $n \longrightarrow \infty$, by the squeeze theorem, $(s_n - t_n)_{n=1}^\infty$ converges to 0.

3. (a) $\lim_{n\to\infty} \frac{1}{n^3} = 0$, since $\frac{1}{n^3} \leq \frac{1}{n}$, and $\frac{1}{n} \longrightarrow 0$ as $n \longrightarrow \infty$.
   (b) $\lim_{n\to\infty} \frac{2n^2-1}{n+1} = \infty$, since $\frac{2n^2-1}{n+1} \geq \frac{2n^2-2}{n+1} = \frac{2(n^2-1)}{n+1} = \frac{2(n-1)(n+1)}{n+1} = 2(n-1)$, and $2(n-1) \longrightarrow \infty$ as $n \longrightarrow \infty$.
   (c) $\lim_{n\to\infty} \frac{3n^2+\sin(n)}{n^2} = 3$, because $\frac{3n^2+\sin(n)}{n^2} = 3 + \frac{\sin(n)}{n^2}$, and by the squeeze theorem, $|\frac{\sin(n)}{n^2}| \leq \frac{1}{n^2}$ implies that $\frac{\sin(n)}{n^2} \longrightarrow 0$ as $\longrightarrow \infty$.
   (d) $\lim_{n\to\infty} \frac{5n^4-2n^2+n+1}{n^2(n^2+1)} = 5$, since $\frac{5n^4-2n^2+n+1}{n^2(n^2+1)} = \frac{5-2n^{-2}+n^{-3}+n^{-4}}{1+n^{-2}}$, and then by a quotient of limits.

4. (a) False: take, for example, $(s_n)_{n=1}^\infty$ such that

$$s_n = \begin{cases} -1 & \text{if } n \text{ even,} \\ 1 & \text{if } n \text{ odd.} \end{cases}$$

   Also, let $t_n = -s_n$. Then $(s_n)_{n=1}^\infty$ and $(t_n)_{n=1}^\infty$ are divergent, but the sequence $(s_n + t_n)_{n=1}^\infty$ (the terms of which are identically 0) is not.

(b) False: consider $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ from part (a). Then $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ again diverge, but the sequence $(s_n \cdot t_n)_{n=1}^{\infty}$ again converges trivially (having terms identically -1).

(c) False: consider $(s_n)_{n=1}^{\infty}$ as above, and $(u_n)_{n=1}^{\infty}$ such that $u_n = 0$ for each n. Then $(s_n)_{n=1}^{\infty}$ and $(s_n + u_n)_{n=1}^{\infty}$ diverge, but $(u_n)_{n=1}^{\infty}$ converges trivially.

(d) False: consider $(\frac{1}{n})_{n=1}^{\infty}$. This sequence converges to 0; however, the sequence $(n)_{n=1}^{\infty}$ diverges to infinity.