Math 25 — Homework Assignment #7

Homework due: **Tuesday 5/17/11** at beginning of discussion section

Reading material. Read sections 2.9, 2.10 in the textbook.

Problems

1. Prove that the sequence

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)}$$

converges. (Note: you are not asked to find what the limit of the sequence is.)

2. Define a sequence $(x_n)_{n=1}^{\infty}$ by

$$x_1 = \sqrt{3}, \quad x_2 = \sqrt{3+\sqrt{3}}, \quad x_3 = \sqrt{3+\sqrt{3+\sqrt{3}}}, \ \ldots \ ,$$
$$x_{n+1} = \sqrt{3+x_n}, \ \ldots$$

Prove that the sequence $(x_n)_{n=1}^{\infty}$ converges, and find its limit. For a small bonus credit, answer the same question for the more general sequence $(y_n)_{n=1}^{\infty}$ defined by

$$y_1 = \sqrt{k}, \qquad y_{n+1} = \sqrt{k+y_n},$$

where k is an arbitary integer ≥ 2 (note that in that case the limit is a function of k).

3. Define the sequence $(a_n)_{n=1}^{\infty}$ by $a_n = \left(1+\frac{1}{n}\right)^n$.

   (a) Show that (a_n) is increasing and bounded from above (see Example 2.36 in section 2.10 of the textbook for ideas) and deduce that it converges. Define the constant e by $e = \lim_{n\to\infty} a_n$.

   (b) Show that $2 < e < 3$.

4. In an analog clock[1], at twelve o'clock both the hours dial and the minutes dial are pointing in the same direction. In this problem, we will find the next time that the two dials are aligned, in two different ways.

   (a) Let X be the number of minutes until the two dials point in the same direction again. Write an algebraic equation that X satisfies, and solve it to find X.

   (b) Another way to compute X is by representing it as the sum of an infinite geometric series, as follows. Define a sequence of times $T_0, T_1, T_2, T_3, \ldots$ by

$$\begin{aligned} T_0 &= \text{twelve o'clock}, \\ T_1 &= \text{the first time after twelve o'clock when the minutes dial} \\ &\qquad \text{returns to the position where the hours dial was at twelve} \\ &\qquad \text{o'clock}, \\ T_2 &= \text{the first time after time } T_1 \text{ when the minutes dial arrives} \\ &\qquad \text{at the position where the hours dial was at time } T_1, \\ &\vdots \\ T_n &= \text{the first time after time } T_{n-1} \text{ when the minutes dial arrives} \\ &\qquad \text{at the position where the hours dial was at time } T_{n-1}, \end{aligned}$$

   etc. For example, it is easy to see that $T_1 =$ 1:00 a.m. and $T_2 =$ 1:05 a.m. Explain why T_n can be written in the form

$$T_n = \text{twelve o'clock } + x_1 + x_2 + \ldots + x_n,$$

   where x_n is the time interval between T_{n-1} and T_n. Find a formula for x_n representing it as a geometric progression. Deduce that

$$X = \lim_{n \to \infty} (x_1 + x_2 + x_3 + \ldots + x_n),$$

   and use the formula

$$1 + a + a^2 + a^3 + \ldots = \frac{1}{1-a}, \qquad (|a| < 1)$$

   for the sum of an infinite geometric series, to compute X in another way.

   (c) Finally, compare the answers you got for X in parts (a) and (b). If they are not the same, you made a mistake somewhere – check your solution and correct it.

[1] If you have never seen an analog clock or don't remember what they look like, see `http://en.wikipedia.org/wiki/Clock_face`