Math 25 — Homework Assignment #8

Homework due: Tuesday 5/24/11 at beginning of discussion section

Reading material. Read sections 2.11, 2.12, 2.13 in the textbook.

Problems

1. For each of the following sequences $(a_n)_{n=1}^\infty$, find $\limsup_{n\to\infty} a_n$, $\liminf_{n\to\infty} a_n$, and the set of subsequential limits of (a_n).

   (a) $a_n = (-1)^n$

   (b) $a_n = \sin\left(\frac{2\pi n}{8}\right)$

   (c) $(a_n)_{n=1}^\infty = \{0, \frac{1}{2}, 1,\ 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1,\ 0, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \ldots, \frac{7}{8}, 1,\ 0, \frac{1}{16}, \ldots, \frac{15}{16}, 1,\ 0, \frac{1}{32}, \ldots\}$

   (d) $a_n = \left(1 + \frac{1}{n}\right)^n$

   (e) $a_n = \left(1 + \frac{1}{n^2}\right)^n$

   Hint: $\left(1 + \frac{1}{n^2}\right)^n = \left[\left(1 + \frac{1}{n^2}\right)^{n^2}\right]^{1/n}$

   (f) $a_n = \left(1 + \frac{1}{n}\right)^{n^2}$

   Hint: $\left(1 + \frac{1}{n}\right)^{n^2} = \left[\left(1 + \frac{1}{n}\right)^n\right]^n$

2. Let $(a_n)_{n=1}^\infty$ be a sequence. Denote $L = \limsup_{n\to\infty} a_n$. Define a sequence $(b_m)_{m=1}^\infty$ by
$$b_m = \sup\{a_m, a_{m+1}, a_{m+2}, \ldots\}.$$

   (a) Prove that b_m is a nonincreasing sequence.
   Hint. Use the fact that if $A, B \subseteq \mathbb{R}$ are sets of real numbers and $A \subseteq B$ then $\sup(A) \le \sup(B)$.

   (b) Prove that $L \le b_m$ for any $m \ge 1$.
   Hint. To make your life a bit easier, prove this first of all for $m = 1$ and then explain why the statement generalizes for all $m \ge 1$.

   (c) Prove that for any $\varepsilon > 0$, there is some m such that $b_m < L + \varepsilon$.

   (d) Use the above results to prove that $L = \lim_{m\to\infty} b_m$.

3. Show that for any bounded sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$,

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Give an example to show that equality need not occur.

4. Prove that a sequence $(a_n)_{n=1}^{\infty}$ converges to a limit L if and only if $L = \lim_{k\to\infty} a_{n_k}$ for every subsequence $(a_{n_k})_{k=1}^{\infty}$.

5. *(Optional problem — not for credit, but recommended).* Previously, we defined an amusing sequence of numbers $(s_n)_{n=1}^{\infty}$ given by

$$s_1 = \sqrt{2},\ s_2 = \sqrt{2+\sqrt{2}},\ s_3 = \sqrt{2+\sqrt{2+\sqrt{2}}},\ \ldots, s_{n+1} = \sqrt{2+s_n},\ \ldots$$

and showed that $\lim_{n\to\infty} s_n = 2$. In this problem we show that this sequence appears naturally in connection with an interesting limiting formula for the mathematical constant π.

(a) Start with the familiar *double-angle identity* from trigonometry:

$$\sin(2x) = 2\sin(x)\cos(x),$$

but rewrite it instead as $\sin(x) = 2\cos\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)$. In this identity, we can use the same identity again to replace $\sin\left(\frac{x}{2}\right)$ on the right-hand side by $2\cos\left(\frac{x}{4}\right)\sin\left(\frac{x}{4}\right)$. This shows that

$$\sin(x) = 4\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\sin\left(\frac{x}{4}\right).$$

Show that by repeating the same trick n times one can eventually arrive at the identity (valid for each $n \ge 1$)

$$\sin(x) = 2^n \cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\cdots\cos\left(\frac{x}{2^n}\right)\sin\left(\frac{x}{2^n}\right).$$

(b) Now we want to use the above identity, which is valid for all real x, with the specific value $x = \pi/2$. To compute $\cos\left(\frac{\pi}{2^k}\right)$, note that we have another double-angle identity,

$$\cos(2x) = \cos^2 x - \sin^x = 2\cos^2 x - 1,$$

which can be rewritten as

$$\cos\left(\frac{x}{2}\right) = \sqrt{\frac{1+\cos x}{2}}, \qquad (\text{if } 0 \le x \le \pi/2).$$

Using this successively starting with $x = \pi/2$ gives

$$\begin{aligned}
\cos(\pi/2) &= 0, \\
\cos(\pi/4) &= \frac{\sqrt{2}}{2}, \\
\cos(\pi/8) &= \frac{\sqrt{2+\sqrt{2}}}{2}, \ldots
\end{aligned}$$

Show that in general we have for all $k \geq 1$ that

$$\cos\left(\frac{\pi}{2^{k+1}}\right) = \frac{s_k}{2}$$

where s_k is defined at the beginning of the question.

(c) Combining the results from the previous parts of the question, show that for all $n \geq 1$,

$$\frac{2}{\pi} = \frac{s_1 s_2 \cdots s_n}{2^n}\left(\sin\left(\frac{\pi}{2^{n+1}}\right) / \left(\frac{\pi}{2^{n+1}}\right)\right)$$

Now use without proof the fact that for any constant a,

$$\lim_{m\to\infty} m \sin\left(\frac{a}{m}\right) = a$$

(this is an easy consequence of the famous limit result from calculus, $\lim\limits_{x\to 0} \dfrac{\sin(x)}{x} = 1$, but here we will assume it without proof), to conclude finally that

$$\frac{2}{\pi} = \lim_{n\to\infty} \frac{s_1 s_2 \cdots s_n}{2^n}.$$

This famous result, proved by the French mathematician Vieta in 1592, is often written in the form of an elegant infinite product of numbers:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2} \cdot \ldots$$