Math 25 — Homework Assignment #8

**Homework due:** Tuesday 5/24/11 at beginning of discussion section

**Reading material.** Read sections 2.11, 2.12, 2.13 in the textbook.

**Problems**

1. For each of the following sequences \((a_n)_{n=1}^\infty\), find \(\limsup_{n \to \infty} a_n\), \(\liminf_{n \to \infty} a_n\), and the set of subsequential limits of \((a_n)\).
   
   (a) \(a_n = (-1)^n\)
   (b) \(a_n = \sin \left( \frac{2\pi n}{5} \right)\)
   (c) \((a_n)_{n=1}^\infty = \{0, \frac{1}{2}, 1, 0, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{8}, \frac{5}{8}, \frac{3}{8}, \frac{7}{8}, 1, 0, \frac{1}{16}, \ldots, \frac{15}{16}, 1, 0, \frac{1}{32}, \ldots \}\)
   (d) \(a_n = \left(1 + \frac{1}{n}\right)^n\)
   (e) \(a_n = \left(1 + \frac{1}{n^2}\right)^n\)
   
   **Hint:** \(\left(1 + \frac{1}{n^2}\right)^n = \left[\left(1 + \frac{1}{n}\right)^{n^2}\right]^{1/n}\)
   (f) \(a_n = \left(1 + \frac{1}{n}\right)^{n^2}\)
   
   **Hint:** \(\left(1 + \frac{1}{n}\right)^{n^2} = \left[\left(1 + \frac{1}{n}\right)^n\right]^n\)

2. Let \((a_n)_{n=1}^\infty\) be a sequence. Denote \(L = \limsup_{n \to \infty} a_n\). Define a sequence \((b_m)_{m=1}^\infty\) by \(b_m = \sup\{a_m, a_{m+1}, a_{m+2}, \ldots\}\).

   (a) Prove that \(b_m\) is a nonincreasing sequence.
   
   **Hint.** Use the fact that if \(A, B \subseteq \mathbb{R}\) are sets of real numbers and \(A \subseteq B\) then \(\sup(A) \leq \sup(B)\).

   (b) Prove that \(L \leq b_m\) for any \(m \geq 1\).
   
   **Hint.** To make your life a bit easier, prove this first of all for \(m = 1\) and then explain why the statement generalizes for all \(m \geq 1\).

   (c) Prove that for any \(\varepsilon > 0\), there is some \(m\) such that \(b_m < L + \varepsilon\).

   (d) Use the above results to prove that \(L = \lim_{m \to \infty} b_m\).
3. Show that for any bounded sequences \((a_n)_{n=1}^{\infty}\) and \((b_n)_{n=1}^{\infty}\),
\[
\limsup_{n \to \infty} (a_n + b_n) \leq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n.
\]

Give an example to show that equality need not occur.

4. Prove that a sequence \((a_n)_{n=1}^{\infty}\) converges to a limit \(L\) if and only if \(L = \lim_{k \to \infty} a_{n_k}\) for every subsequence \((a_{n_k})_{k=1}^{\infty}\).

5. (Optional problem — not for credit, but recommended). Previously, we defined an amusing sequence of numbers \((s_n)_{n=1}^{\infty}\) given by
\[
s_1 = \sqrt{2}, \quad s_2 = \sqrt{2 + \sqrt{2}}, \quad s_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \ldots, \quad s_{n+1} = \sqrt{2 + s_n}, \ldots
\]
and showed that \(\lim_{n \to \infty} s_n = 2\). In this problem we show that this sequence appears naturally in connection with an interesting limiting formula for the mathematical constant \(\pi\).

(a) Start with the familiar double-angle identity from trigonometry:
\[
\sin(2x) = 2 \sin(x) \cos(x),
\]
but rewrite it instead as \(\sin(x) = 2 \cos\left(\frac{x}{2}\right) \sin\left(\frac{x}{2}\right)\). In this identity, we can use the same identity again to replace \(\sin\left(\frac{x}{2}\right)\) on the right-hand side by \(2 \cos\left(\frac{x}{4}\right) \sin\left(\frac{x}{4}\right)\). This shows that
\[
\sin(x) = 4 \cos\left(\frac{x}{2}\right) \cos\left(\frac{x}{4}\right) \sin\left(\frac{x}{4}\right).
\]
Show that by repeating the same trick \(n\) times one can eventually arrive at the identity (valid for each \(n \geq 1\))
\[
\sin(x) = 2^n \cos\left(\frac{x}{2}\right) \cos\left(\frac{x}{4}\right) \cos\left(\frac{x}{8}\right) \cdots \cos\left(\frac{x}{2^n}\right) \sin\left(\frac{x}{2^n}\right).
\]

(b) Now we want to use the above identity, which is valid for all real \(x\), with the specific value \(x = \pi/2\). To compute \(\cos\left(\frac{\pi}{2^k}\right)\), note that we have another double-angle identity,
\[
\cos(2x) = \cos^2(x) - \sin^2(x) = 2 \cos^2(x) - 1,
\]
which can be rewritten as
\[
\cos\left(\frac{x}{2}\right) = \sqrt{\frac{1 + \cos x}{2}}, \quad \text{(if } 0 \leq x \leq \pi/2\).
\]
Using this successively starting with \( x = \pi/2 \) gives
\[
\cos(\pi/2) = 0,
\]
\[
\cos(\pi/4) = \frac{\sqrt{2}}{2},
\]
\[
\cos(\pi/8) = \frac{\sqrt{2 + \sqrt{2}}}{2}, \ldots
\]
Show that in general we have for all \( k \geq 1 \) that
\[
\cos\left(\frac{\pi}{2^k+1}\right) = \frac{s_k}{2}
\]
where \( s_k \) is defined at the beginning of the question.

(c) Combining the results from the previous parts of the question, show that for all \( n \geq 1 \),
\[
\frac{2}{\pi} = \frac{s_1 s_2 \cdots s_n}{2^n} \left( \sin\left(\frac{\pi}{2^{n+1}}\right) / \left(\frac{\pi}{2^{n+1}}\right) \right)
\]
Now use without proof the fact that for any constant \( a \),
\[
\lim_{m \to \infty} m \sin\left(\frac{a}{m}\right) = a
\]
(this is an easy consequence of the famous limit result from calculus, \( \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \), but here we will assume it without proof),
to conclude finally that
\[
\frac{2}{\pi} = \lim_{n \to \infty} \frac{s_1 s_2 \cdots s_n}{2^n}.
\]
This famous result, proved by the French mathematician Vieta in 1592, is often written in the form of an elegant infinite product of numbers:
\[
\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2 + \sqrt{2}}}{2} \cdot \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \cdot \frac{\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}{2} \cdot \ldots
\]