Math 25 – Solutions to Homework Assignment #8

1. (a) $\limsup_{n\to\infty} a_n = 1$; $\liminf_{n\to\infty} a_n = -1$; the set of subsequential limits is $\{-1, 1\}$.

   (b) $\limsup_{n\to\infty} a_n = 1$; $\liminf_{n\to\infty} a_n = -1$; the set of subsequential limits is $\{-1, -2^{-\frac{1}{2}}, 0, 2^{-\frac{1}{2}}, 1\}$.

   (c) $\limsup_{n\to\infty} a_n = 1$; $\liminf_{n\to\infty} a_n = -1$; the set of subsequential limits is $[0, 1]$.

   (d) $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = e$; the set of subsequential limits is $\{e\}$.

   (e) $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = 1$; the set of subsequential limits is $\{1\}$.

   (f) $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \infty$; the set of subsequential limits is $\varnothing$.

2. (a) For any $m \in \mathbb{N}$, since $\{a_{m+1}, a_{m+2}, \ldots\} \subseteq \{a_m, a_{m+1}, a_{m+2}, \ldots\}$, $b_{m+1} = \sup\{a_{m+1}, a_{m+2}, \ldots\} \leq \sup\{a_m, a_{m+1}, a_{m+2}, \ldots\} = b_m$. Thus, $(b_m)_{m=1}^{\infty}$ is nonincreasing.

   (b) Since (from above) $(b_m)_{m=1}^{\infty}$ is nonincreasing, either there exists some k such that $b_n = \alpha$ for some α and all $n \geq k$, or not. If this is the case, then $\lim_{n\to\infty} b_n = L = \alpha$, so that $b_m \geq L$ for all m. If not, then for each m, there exists an M such that $b_n < b_m$ for all $n \geq M$. Thus, for each m, $b_m \in \{\beta : \exists j \text{ s.t. } b_i < \beta \text{ if } i \geq j\}$ (by monotonicity), so that $b_m \geq \inf\{\beta : \exists j \text{ s.t. } b_i < \beta \text{ if } i \geq j\} = L$.

   (c) Assume, by way of contradiction, that there exists some $\epsilon > 0$ such that, for all m, $b_m \geq L+\epsilon$. Then for each β such that $b_n < \beta$ for some n, $\beta \geq L+\epsilon$, and it follows that $L = \inf\{\beta : \exists j \text{ s.t. } b_i < \beta \text{ if } i \geq j\} \geq L+\epsilon$. Then $\epsilon \leq 0$, a contradiction.

   (d) By the results from (b), (a), and (c) above, respectively, for any $\epsilon > 0$, there exists some m such that $|b_n - L| = b_n - L \leq b_m - L < \epsilon$ whenever $n \geq m$.

3. Above, we have found that $\limsup_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} \sup_{m\geq n} (a_m + b_m)$. If $(a_m)_{m=1}^{\infty}$ and $(b_m)_{m=1}^{\infty}$ are bounded, then for each n,

$$\sup_{m\geq n} (a_m + b_m) \leq \sup_{m\geq n} a_m + \sup_{m\geq n} b_m$$

so that

$$\limsup_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} \sup_{m\geq n} (a_m + b_m)$$

$$\leq \lim_{n\to\infty} \left(\sup_{m\geq n} a_m + \sup_{m\geq n} b_m \right) = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Equality need not hold if, for example, for each n, $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$; in this case, $\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n = 1$, but $\limsup_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} (a_n + b_n) = 0$.

4. That $(a_n)_{n=1}^{\infty}$ converges to a limit L if so do its subsequences $(a_{n_k})_{k=1}^{\infty}$ follows immediately, since the sequence is a subsequence of itself. In the other direction, Let $a_n \underset{n\to\infty}{\longrightarrow} L$. For any subsequence $(a_{n_k})_{k=1}^{\infty}$, fix some $\epsilon > 0$. By definition, there exists some m such that $|a_n - L| < \epsilon$ if $n \geq m$. For $M = \min\{k : n_k \geq m\}$, then, $|a_{n_i} - L| < \epsilon$ for $i \geq M$; so, by definition, $\lim_{k\to\infty} a_{n_k} = L$.

5. (a) We could argue by induction: firstly, note that substitution of $\frac{x}{2}$ in place of x in the double-angle identity yields $\sin(x) = 2\sin(\frac{x}{2})\cos(\frac{x}{2})$. Now, another substitution yields, for any n, $\sin(\frac{x}{2^n}) = 2\sin(\frac{x}{2^{n+1}})\cos(\frac{x}{2^{n+1}})$; so, if

$$\sin(x) = 2^n \cos\left(\frac{x}{2}\right) \cos\left(\frac{x}{4}\right) \cdots \cos\left(\frac{x}{2^n}\right) \sin\left(\frac{x}{2^n}\right)$$

then we have

$$\sin(x) = 2^{n+1} \cos\left(\frac{x}{2}\right) \cos\left(\frac{x}{4}\right) \cdots \cos\left(\frac{x}{2^n}\right) \cos\left(\frac{x}{2^{n+1}}\right) \sin\left(\frac{x}{2^{n+1}}\right)$$

so that this identity is valid for all $n \geq 1$.

(b) Since, for $x \in [0, \frac{\pi}{2}]$,

$$\cos\left(\frac{x}{2}\right) = \sqrt{\frac{1+\cos x}{2}},$$

we have $\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} = \frac{s_1}{2}$. Moreover, if $\cos(\frac{\pi}{2^{k+1}}) = \frac{s_k}{2}$, then

$$\cos\left(\frac{\pi}{2^{k+2}}\right) = \sqrt{\frac{1+\frac{s_k}{2}}{2}} = \frac{\frac{1}{\sqrt{2}}\sqrt{2+s_k}}{\sqrt{2}} = \frac{s_{k+1}}{2},$$

so that $\cos(\frac{\pi}{2^{n+1}}) = \frac{s_n}{2}$ for each $n \geq 1$, by induction.

(c) Putting together parts (a) and (b), for any n,

$$1 = \sin\left(\frac{\pi}{2}\right) = 2^n \cos\left(\frac{\pi}{2^2}\right) \cos\left(\frac{\pi}{2^3}\right) \cdots \cos\left(\frac{\pi}{2^{n+1}}\right) \sin\left(\frac{\pi}{2^{n+1}}\right)$$

$$= 2^n \left(\frac{s_1 s_2 \cdots s_n}{2^n} \right) \sin\left(\frac{\pi}{2^{n+1}} \right)$$

so that

$$\begin{aligned} \frac{2}{\pi} &= 2^n \left(\frac{s_1 s_2 \cdots s_n}{2^n} \right) \sin\left(\frac{\pi}{2^{n+1}} \right) \left(\frac{2}{\pi} \right) \\ &= \left(\frac{s_1 s_2 \cdots s_n}{2^n} \right) \left(\sin\left(\frac{\pi}{2^{n+1}} \right) / \frac{\pi}{2^{n+1}} \right) \end{aligned}$$

Since $\lim_{n\to\infty} 2^{n+1} \sin\left(\frac{\pi}{2^{n+1}} \right) = \pi$, it follows that

$$\begin{aligned} \lim_{n\to\infty} \frac{s_1 s_2 \ldots s_N}{2^n} &= \lim_{n\to\infty} \frac{2}{\pi} \cdot \frac{\pi/2^{n+1}}{\sin(\pi/2^{n+1})} \\ &= \frac{2}{\pi} \lim_{n\to\infty} \frac{\pi/2^{n+1}}{\sin(\pi/2^{n+1})} = \frac{2}{\pi}. \end{aligned}$$