1. Decide whether or not each of the following infinite series converges or diverges. Prove your claims.

\[
\begin{align*}
\text{(a)} & \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} \\
\text{(b)} & \quad \sum_{n=1}^{\infty} \frac{1}{n^n} \\
\text{(c)} & \quad \sum_{n=1}^{\infty} \frac{n(n + 1)}{(n + 2)^2} \\
\text{(d)} & \quad \sum_{n=1}^{\infty} \frac{3n(n + 1)(n + 2)}{n^3\sqrt{n}} \\
\text{(e)} & \quad \sum_{n=1}^{\infty} \frac{n + 2}{(n^2 + 1)\sqrt{n}} \\
\text{(f)} & \quad \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}
\end{align*}
\]

2. Find the value of the (hint: telescoping) infinite sum

\[
\sum_{n=1}^{\infty} \frac{1}{n(n + 1)(n + 2)} = \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \frac{1}{4 \cdot 5 \cdot 6} + \ldots
\]

3. If \((a_n)_{n=1}^{\infty}\) and \((b_n)_{n=1}^{\infty}\) are sequences such that \(a_n, b_n \geq 0\) for all \(n \geq 1\), and the sums \(\sum_{n=1}^{\infty} a_n\) and \(\sum_{n=1}^{\infty} b_n\) are convergent, prove that \(\sum_{n=1}^{\infty} a_n b_n\) converges as well.

4. In this problem we study the convergence of the series \(\sum_{n=1}^{\infty} a_n\), where

\[
a_n = \frac{(n!)^2}{(2n)!} = \frac{1}{\binom{2n}{n}}.
\]

(a) Find a simple formula for \(\frac{a_{n+1}}{a_n}\), and show that \(\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{1}{4}\).

(b) Deduce that for any \(\varepsilon > 0\), there is a constant \(C > 0\) such that \(a_n \leq C(\frac{1}{4} + \varepsilon)^n\).

(c) Use the comparison test to deduce that the series converges.

**Hint.** Sections 3.6.4 and 3.6.5 in the textbook contain a general discussion that would give you some guidance for this problem.
(d) *(Optional)* An alternative approach to showing convergence of the series rests on the fact that \( \frac{1}{a_n} = \binom{2n}{n} \) is a binomial coefficient. Observe that by the binomial theorem,

\[
4^n = (1 + 1)^{2n} = 1 + 2n + \frac{2n(2n - 1)}{2} + \frac{2n(2n - 1)(2n - 2)}{6} + \ldots + \binom{2n}{k} + \ldots + \binom{2n}{2n}.
\]

That is, the sum of the coefficients in the \((2n)\)th row of Pascal’s triangle is \(4^n\). In particular, \(\binom{2n}{n} \leq 4^n\), so \(a_n \geq \frac{1}{4^n}\). Unfortunately, to prove convergence of \(\sum_{n=1}^{\infty} a_n\) by comparison of series we need a bound for \(a_n\) from *above* rather than from below. To derive such a bound, argue that \(\binom{2n}{n}\), the middle coefficient in the \((2n)\)th row of Pascal’s triangle, is larger than all the other coefficients on the same row (i.e., that \(\binom{2n}{k} \geq \binom{2n}{k}\) for all \(0 \leq k \leq 2n\)), and conclude from this that \(a_n \leq (2n + 1)\frac{1}{4^n}\). Then explain why this is enough to deduce convergence.

**Note.** In *really* advanced calculus, one learns how to compute the value of such sums. A rather nontrivial result says that this one is equal to

\[
\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}} = \frac{1}{3} + \frac{2\pi\sqrt{3}}{27}.
\]

5. Prove that the series

\[
\sum_{n=1}^{\infty} \frac{1}{n \log_2 n}
\]

diverges (where \(\log_2 n\) is the base-2 logarithm of \(n\) — you may replace it with \(\ln n\) if you prefer, since \(\log_2 n = \ln(n)/\ln 2\)). Note that the terms being summed converge to 0 faster than \(1/n\), but slower than \(1/n^p\) for \(p > 1\), so the behavior of this series lies precisely on the boundary between convergence and divergence in the generalized harmonic series and so is too fine to be determined using a simple comparison.

**Hint.** Show that the partial sums are unbounded, using a similar technique to the one used to prove that the harmonic series diverges.

6. Alter the harmonic series \(\sum_{n=1}^{\infty} \frac{1}{n}\) by deleting all terms in which the denominator contains the digit 9. Show that the new series converges.

**Hint.** See Note 52 at the end of Chapter 3.