# Math 25 — Homework Assignment #9

**Homework due:** **Tuesday 5/31/11** at beginning of discussion section

**Reading material.** Read section 3.4, 3.5, 3.6.1, 3.6.2, 3.6.12 in the textbook.

**Problems**

1. Decide whether or not each of the following infinite series converges or diverges. Prove your claims.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{3n(n+1)(n+2)}{n^3\sqrt{n}}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$ (e) $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{(n^2+1)\sqrt{n}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)^2}$ (f) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$

2. Find the value of the (**hint:** telescoping) infinite sum
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} + \frac{1}{4\cdot 5\cdot 6} + \dots$$

3. If $(a_n)_{n=1}^{\infty}$ $(b_n)_{n=1}^{\infty}$ are sequences such that $a_n, b_n \ge 0$ for all $n \ge 1$, and the sums $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent, prove that $\sum_{n=1}^{\infty} a_n b_n$ converges as well.

4. In this problem we study the convergence of the series $\sum_{n=1}^{\infty} a_n$, where
$$a_n = \frac{(n!)^2}{(2n)!} = \frac{1}{\binom{2n}{n}}.$$

   (a) Find a simple formula for $\frac{a_{n+1}}{a_n}$, and show that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{4}$.

   (b) Deduce that for any $\varepsilon > 0$, there is a constant $C > 0$ such that $a_n \le C(\frac{1}{4}+\varepsilon)^n$.

   (c) Use the comparison test to deduce that the series converges.

   **Hint.** Sections 3.6.4 and 3.6.5 in the textbook contain a general discussion that would give you some guidance for this problem.

(d) *(Optional)* An alternative approach to showing convergence of the series rests on the fact that $1/a_n = \binom{2n}{n}$ is a binomial coefficient. Observe that by the binomial theorem,

$$\begin{aligned} 4^n =& (1+1)^{2n} = 1 + 2n + \frac{2n(2n-1)}{2} + \frac{2n(2n-1)(2n-2)}{6} \\ &+ \ldots + \binom{2n}{k} + \ldots + \binom{2n}{2n}. \end{aligned}$$

That is, the sum of the coefficients in the $(2n)$th row of Pascal's triangle is $4^n$. In particular, $\binom{2n}{n} \leq 4^n$, so $a_n \geq \frac{1}{4^n}$. Unfortunately, to prove convergence of $\sum_{n=1}^{\infty} a_n$ by comparison of series we need a bound for $a_n$ *from above* rather than from below. To derive such a bound, argue that $\binom{2n}{n}$, the middle coefficient in the $(2n)$th row of Pascal's triangle, is larger than all the other coefficients on the same row (i.e., that $\binom{2n}{n} \geq \binom{2n}{k}$ for all $0 \leq k \leq 2n$), and conclude from this that $a_n \leq (2n+1)\frac{1}{4^n}$. Then explain why this is enough to deduce convergence.

**Note.** In *really* advanced calculus, one learns how to compute the value of such sums. A rather nontrivial result says that this one is equal to

$$\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}} = \frac{1}{3} + \frac{2\pi\sqrt{3}}{27}.$$

5. Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n \log_2 n}$$

diverges (where $\log_2 n$ is the base-2 logarithm of $n$ — you may replace it with $\ln n$ if you prefer, since $\log_2 n = \ln(n)/\ln 2$). Note that the terms being summed converge to 0 faster than $1/n$, but slower than $1/n^p$ for $p > 1$, so the behavior of this series lies precisely on the boundary between convergence and divergence in the generalized harmonic series and so is too fine to be determined using a simple comparison.

**Hint.** Show that the partial sums are unbounded, using a similar technique to the one used to prove that the harmonic series diverges.

6. Alter the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ by deleting all terms in which the denominator contains the digit 9. Show that the new series converges.

**Hint.** See Note 52 at the end of Chapter 3.