Math 25 — Solutions to Homework Assignment #9

1. Decide whether or not each of the following infinite series converges or diverges. Prove your claims.

   (a) The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$ diverges.

   Proof. Consider the sequence $a_n = \sqrt[n]{n}$. Note that

   $$\lim_{n\to\infty} \log\left(\sqrt[n]{n}\right) = \lim_{n\to\infty} \frac{\log n}{n} = 0,$$

   which implies $a_n \to 1$. So, by the nth term test (also called the trivial test in the textbook), the series diverges. □

   (b) The series $\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges.

   Proof. Note that $n^k \le n^j$ for $n, k, j \in \mathbb{N}$ and $k < j$, so that $\frac{1}{n^k} \ge \frac{1}{n^j}$. Then,

   $$\sum_{n=1}^{\infty} \frac{1}{n^n} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^n} \le 1 + \sum_{n=2}^{\infty} \frac{1}{n^2},$$

   so the series converges by the comparison test. □

   (c) The series $\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)^2}$ diverges.

   Proof. Note that

   $$\lim_{n\to\infty} \frac{n(n+1)}{(n+2)^2} = 1,$$

   so the series diverges by the nth term test. □

   (d) The series $\sum_{n=1}^{\infty} \frac{3n(n+1)(n+2)}{n^3\sqrt{n}}$ diverges.

   Proof. Note that

   $$\sum_{n=1}^{\infty} \frac{3n(n+1)(n+2)}{n^3\sqrt{n}} = \sum_{n=1}^{\infty} \frac{3n^3 + 9n^2 + 6n}{n^3\sqrt{n}} = \sum_{n=1}^{\infty} \frac{3n^3}{n^3\sqrt{n}} + \frac{9n^2}{n^3\sqrt{n}} + \frac{6n}{n^3\sqrt{n}} \ge 3\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}},$$

   which diverges by the p-series test. So, by the comparison test, the original series diverges. □

(e) The series $\sum_{n=1}^{\infty} \frac{n+2}{(n^2+1)\sqrt{n}}$ converges.

Proof. Note that

$$\sum_{n=1}^{\infty} \frac{n+2}{(n^2+1)\sqrt{n}} = \sum_{n=1}^{\infty} \frac{n}{(n^2+1)\sqrt{n}} + \frac{2}{(n^2+1)\sqrt{n}} \leq \sum_{n=1}^{\infty} \frac{n}{(n^2+1)\sqrt{n}} \leq \sum_{n=1}^{\infty} \frac{n}{(n^2)\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}},$$

which converges by the p-series test. So, by the comparison test, the original series converges. □

(f) The series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ converges.

Proof. Note that the sequence $a_n = \ln(n)$ is monotonically increasing and unbounded. Then $\frac{1}{\ln(n)}$ is monotonically decreasing to 0. So the series converges by the alternating series test. □

2. Find the value of the telescoping infinite sum

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}.$$

Proof. Note that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \sum_{n=1}^{\infty} \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2n+4}.$$

Furthermore, note that

$$\frac{1}{2(n+2)} - \frac{1}{(n+1)+1} + \frac{1}{2n+4} = 0.$$

From this, we see that the terms of the form $\frac{1}{2n}$ will cancel for $n \geq 3$, the terms of the form $-\frac{1}{n+1}$ will cancel for $n \geq 2$, and all of the terms of the form $\frac{1}{2n+4}$ will cancel. That is, the only terms that do not cancel are

$$\frac{1}{2} - \frac{1}{2} + \frac{1}{4} = \frac{1}{4}.$$

So we conclude that the series converges to $\frac{1}{4}$. Alternatively, by writing out the partial sums, one can see that

$$S_n = \frac{1}{4} + \frac{1}{2(n+1)} - \frac{1}{n+1} + \frac{1}{2n+4},$$

which clearly converges to $\frac{1}{4}$. □

3. If $\{a_n\}$ and $\{b_n\}$ are sequences such that $a_n, b_n \geq 0$ and the sums $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, prove that $\sum_{n=1}^{\infty} a_n b_n$ converges.

Proof. Consider the partial sums

$$S_k = \sum_{n=1}^{k} a_n \qquad \text{and} \qquad T_k = \sum_{n=1}^{k} b_n.$$

Then, we know that the sequence $S_k T_k$ converges. Now,

$$S_k T_k = \sum_{n=1}^{k} a_n \cdot \sum_{n=1}^{k} b_n = a_1 \sum_{n=1}^{k} b_n + a_2 \sum_{n=1}^{k} b_n + \cdots + a_k \sum_{n=1}^{k} b_n \geq a_1 b_1 + a_2 b_2 + \cdots + a_k b_k = \sum_{n=1}^{k} a_n b_n,$$

where the inequality is justified because $a_n, b_n \geq 0$. Then, since the sequence of partial sums is monotonically increasing and bounded from above, it must converge. □

4. Consider the series

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}.$$

(a) Find a simple formula for $\frac{a_{n+1}}{a_n}$, and show that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{4}$.

Proof. Note that

$$\frac{a_{n+1}}{a_n} = \frac{[(n+1)!]^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} = \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{n^2+2n+1}{4n^2+6n+2},$$

and it is easy to see that this converges to $\frac{1}{4}$. □

(b) Deduce that for any $\varepsilon > 0$, there is a constant $C > 0$ such that $a_n \leq C\left(\frac{1}{4}+\varepsilon\right)^n$.

Proof. Fix $\varepsilon > 0$. Since as was shown above $a_{n+1}/a_n \to 1/4$ as $n \to \infty$, it follows that there exists an $N \geq 1$ such that for all $n \geq N$, $a_{n+1}/a_n < 1/4 + \varepsilon$. It follows that for $n \geq N$,

$$a_n = \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdot \frac{a_{n-2}}{a_{n-3}} \cdot \ldots \cdot \frac{a_{N+2}}{a_{N+1}} \cdot \frac{a_{N+1}}{a_N} \cdot a_N \leq a_N \left(\tfrac{1}{4}+\varepsilon\right)^{n-N} = \frac{a_N}{\left(\frac{1}{4}+\varepsilon\right)^N} \left(\tfrac{1}{4}+\varepsilon\right)^n$$

So, if we define $C = a_N / \left(\frac{1}{4}+\varepsilon\right)^N$ then the inequality $a_n \leq C\left(\frac{1}{4}+\varepsilon\right)^n$ holds for $n \geq N$. By making C bigger if necessary we can also make sure that the inequality holds for $n = 1, 2, \ldots, N-1$ (just take C bigger than the maximum of the ratios $a_n/(1/4+\varepsilon)^n$ over $n = 1, \ldots, N-1$). □

(c) Use the comparison test to deduce that the series converges.

Proof. We have bounded the original series above by a geometric series, so the original series converges. □

5. Prove that the series

$$\sum_{n=2}^{\infty} \frac{1}{n \log_2 n}$$

diverges.

Proof. The easiest way to solve this problem is to apply the integral test. Doing so, we obtain,

$$\int_2^{\infty} \frac{1}{x \log_2 x}\, dx = \int_2^{\infty} \frac{1}{t}\, dt,$$

which clearly diverges. Alternatively, one may instead examine the sequence of partial sums of the given series. Doing so, we obtain

$$\begin{aligned} S_k &= \frac{1}{2\log_2 2} + \frac{1}{3\log_2 3} + \cdots + \frac{1}{k\log_2 k} \geq \frac{1}{2} + \frac{1}{6} + \frac{1}{8} + \frac{1}{15} + \frac{1}{18} + \frac{1}{21} + \frac{1}{24} + \frac{1}{36} + \cdots \\ &\geq \frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{24} + \frac{1}{24} + \frac{1}{24} + \frac{1}{24} + \cdots = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots, \end{aligned}$$

from which we see the series diverges. □

6. Alter the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ by deleting all terms in which the denominator contains the digit 9. Show that the new series converges.

Proof. According to the hint, the number of elements of the series that have k digits in the denominator is equal to $8 \cdot 9^{k-1}$. Note that the first (and largest) element of the series that has k digits in the denominator is $\frac{1}{10^{k-1}}$. Then, if we write the series in the form

$$\sum_{n=1}^{\infty} a_n = T_1 + T_2 + T_3 + \cdots$$

where T_k is the sum of all fractions on our list with k digits in the denominator. Then we see that $T_k \leq \frac{1}{10^{k-1}} 8 \cdot 9^{k-1}$, which forms a geometric series with $r < 1$, so by the comparison test, the original series converges. □