Math 25 — Solutions to practice problems

Question 1

For $n = 0, 1, 2, 3, \ldots$ and $0 \le k \le n$ define numbers C_k^n by

$$C_k^n = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\ldots(n-k+1)}{k!}$$

(for $k = 0$ and $k = n$ we define $C_0^n = C_n^n = 1$).

(a) Prove that for all $n \ge 1$ and $1 \le k \le n-1$,

$$C_k^n = C_{k-1}^{n-1} + C_k^{n-1}$$

Proof.

$$\begin{aligned} C_{k-1}^{n-1} + C_k^{n-1} &= \frac{(n-1)(n-2)\ldots(n-k+1)}{(k-1)!} + \frac{(n-1)(n-2)\ldots(n-k)}{k!} \\ &= \frac{k+(n-k)}{k} \cdot \frac{(n-1)\ldots(n-k+1)}{(k-1)!} \\ &= \frac{n}{k} \cdot \frac{(n-1)\ldots(n-k+1)}{(k-1)!} = C_k^n \end{aligned}$$

□

(b) Prove by induction on n that for all $n \ge 1$,

$$\sum_{k=0}^{n} C_k^n = C_0^n + C_1^n + C_2^n + \ldots + C_n^n = 2^n$$

Proof. For $n = 1$ we have $C_0^n + C_1^n = 1 + 1 = 2^1$, so the claim is true. Let $n \ge 1$ be given, and assume the claim is true for that value of n. Then, by using the result of (a) above, we see that

$$\begin{aligned} \sum_{k=0}^{n+1} C_k^{n+1} &= 1 + \sum_{k=1}^{n} C_k^{n+1} + 1 = 2 + \sum_{k=1}^{n} \left(C_{k-1}^n + C_k^n \right) \\ &= 2 + \sum_{j=0}^{n} C_j^n + \sum_{k=1}^{n} C_k^n = 2 + \left(\sum_{j=0}^{n-1} C_j^n - 1 \right) + \left(\sum_{k=0}^{n-1} C_k^n - 1 \right) \\ &= 2 + (2^n - 1) + (2^n - 1) = 2 \cdot 2^n = 2^{n+1}, \end{aligned}$$

where we used the inductive hypothesis in the transition from the second row to the third. This shows that the claim is true for $n+1$ and completes the induction.

$\square$

Question 2

(a) Let $(a_n)_{n=1}^\infty, (b_n)_{n=1}^\infty$ be sequences of real numbers. For each of the following identities, explain what assumptions are needed to ensure that the identity is valid:

i. $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

ii. $\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

iii. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$

Solution. By a theorem we learned, i. and ii. are valid under the assumption that the sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are convergent. For iii., one needs the additional assumption that the limit of $(b_n)_{n=1}^\infty$ is not zero.

(b) Find the limit L of the sequence given by

$$a_n = \frac{5n^4 + 3n^2 - 10}{(2n^2 + \sin(n))^2}$$

Prove rigorously that $L = \lim_{n\to\infty} a_n$, by appealing either to the definition of the limit or to known results about limits.

Solution. First, we bring a_n to a form more suitable for computing the limit, by noting that

$$a_n = \frac{5 + \frac{3}{n^2} - \frac{10}{n^4}}{\left(2 + \frac{\sin(n)}{n^2}\right)^2}.$$

Now we apply the theorem referred to in part (a) above, with the standard limits

$$\begin{aligned}
\lim_{n\to\infty} 5 &= 5, \\
\lim_{n\to\infty} \frac{3}{n^2} &= 0, \\
\lim_{n\to\infty} \frac{-10}{n^4} &= 0, \\
\lim_{n\to\infty} 2 &= 2, \\
\lim_{n\to\infty} \frac{\sin(n)}{n^2} &= 0.
\end{aligned}$$

The last limit is a result of an application of the squeeze theorem, since we have the inequalities

$$\frac{-1}{n^2} \leq \frac{\sin(n)}{n^2} \leq \frac{1}{n^2}$$

Putting these results together, we conclude that $(a_n)_{n=1}^{\infty}$ is convergent, and its limit is equal to

$$\lim_{n\to\infty} a_n = \frac{5+0-0}{(2+0)(2+0)} = \frac{5}{4}.$$

Question 3

(a) State the squeeze theorem.

Solution. See Theorem 2.20 in Section 2.8 in the textbook.

(b) Denote $a_n = \sqrt[n]{n}$. Prove that $\lim_{n\to\infty} a_n = 1$. You may use the fact that the inequality $(1+x)^n \geq \frac{n(n-1)}{2}x^2$ holds for all $n \geq 1$ and $x > 0$.

Solution. See Example 2.33 in Section 2.10 in the textbook.

Question 4

(a) Prove that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

(b) Let p be a positive real number. Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

(c) Prove that the series $\sum_{n=2}^{\infty} \frac{1}{n \log_2 n}$ diverges (here, $\log_2 n = \ln(n)/\ln(2)$ denotes the base-2 logarithm of n).

Solution. For parts (a) and (b) see Section 3.4.2 in the textbook. For part (c), see the solution to problem 5 in homework assignment #9.

Question 5

(a) Evaluate the infinite series

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \ldots$$

Solution. Recall the formula for the sum of a geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

which is valid for all $-1 < x < 1$. The above series is a geometric series with $x = -\frac{1}{2}$, so its sum is

$$\frac{1}{1 - \left(-\frac{1}{2}\right)} = \frac{1}{3/2} = \frac{2}{3}.$$

(b) Evaluate the infinite series

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots$$

Solution. This is a geometric series with $x = 1/3$, so by the formula above its sum is $1/(1 - 1/3) = 3/2$.

(c) Evaluate the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \frac{1}{7\cdot 9} + \dots$$

Solution. Note that

$$\frac{1}{4n^2-1} = \frac{1}{(2n-1)(2n+1)} = \tfrac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right).$$

So, this is a telescoping sum:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{4n^2-1} &= \tfrac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &= \tfrac{1}{2}\left(\tfrac{1}{1} - \tfrac{1}{3} + \tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{5} - \tfrac{1}{7} + \dots\right) = \tfrac{1}{2}.\end{aligned}$$

(Note: to properly prove convergence one needs to do this computation more carefully for the partial sums first; but, here the question only asked to find the value of the series).

Question 6

Give an example of:

(a) A divergent series $\rightarrow \sum_{n=1}^{\infty} 1$

(b) An absolutely convergent series $\rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n}$

(c) A conditionally convergent series $\rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

(d) A series of the form $\sum_{n=1}^{\infty}(a_n + b_n)$ that is convergent but such that both series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent.

Solution. Take $a_n = 1$, $b_n = -1$

(e) A series of the form $\sum_{n=1}^{\infty}(a_n \cdot b_n)$ that is convergent but such that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent.

Solution. Take $a_n = b_n = \frac{1}{n}$.

(f) A divergent series with bounded partial sums $\quad \rightarrow \sum_{n=1}^{\infty}(-1)^n$

(g) A series $\sum_{n=1}^{\infty} a_n$ that is divergent such that $\lim_{n\to\infty}(n \cdot a_n) = 0$ (i.e., loosely speaking, the sequence being summed converges to 0 faster than $1/n$).

Solution. The series $\sum_{n=2}^{\infty} \frac{1}{n \log_2 n}$, which appears in question 4(c) above.

Question 7

(a) Prove that $\lim_{n\to\infty} \frac{n}{2^n} = 0$. You may use the fact that the inequality $(1 + x)^n \geq \frac{n(n-1)}{2}x^2$ holds for all $n \geq 1$ and $x > 0$.

Solution. Taking $x = 1$ in the inequality mentioned in the question gives

$$2^n \geq \frac{n(n-1)}{2},$$

so, for $n \geq 2$,

$$0 \leq \frac{n}{2^n} \leq \frac{2}{n-1},$$

and the claim follows from the squeeze theorem.

(b) Prove that $\sum_{n=1}^{\infty} \frac{1}{2^n - n}$ converges.

Solution. Since $\lim_{n\to\infty} \frac{n}{2^n} = 0$ as shown above, taking $\varepsilon = \frac{1}{10}$ in the definition of the limit, we get that there is some N such that

$$\frac{n}{2^n} \leq \frac{1}{10}$$

for all $n \geq N$. Therefore, for $n \geq N$,

$$\frac{1}{2^n - n} = \frac{1}{2^n\left(1 - \frac{n}{2^n}\right)} \leq \frac{1}{2^n\left(1 - \frac{1}{10}\right)} = \frac{10}{9} \cdot \frac{1}{2^n}.$$

Since also $2^n \geq 10n$, so $1/(2^n - n) > 0$, it follows that $\displaystyle\sum_{n=N}^{\infty} \frac{1}{2^n - n}$ converges, by the comparison test. This is different than the series given in the question in that the summation starts at $n = N$ instead of $n = 1$, but as we know, the convergence of a series only depends on its "tail" — i.e., omitting or adding only a finite number of initial terms does not affect the convergence, and therefore the original series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - n}$ also converges.

Question 8

The goal of this problem is to compute the value of the infinite sum

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \ldots$$

(and in particular to show that it converges, which strengthens the result of question 7(a) above).

(a) Define a new sequence $(x_n)_{n=1}^{\infty}$ whose terms are given by

$$(x_n)_{n=1}^{\infty} = \left\{ \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{32}, \tfrac{1}{32}, \tfrac{1}{32}, \tfrac{1}{32}, \tfrac{1}{32}, \ldots \right\}$$

Explain why if $\displaystyle\sum_{n=1}^{\infty} x_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$ also converges and the values of the two series are the same.

Solution. It is easy to see that the sequence of partial sums of $\sum_{n=1}^{\infty} \frac{n}{2^n}$ is a subsequence of the sequence of partial sums of $\sum_{n=1}^{\infty} x_n$. More precisely, if we denote

$$s_n = \sum_{k=1}^{n} \frac{k}{2^k}, \qquad t_n = \sum_{k=1}^{n} x_k,$$

then we have

$$\begin{aligned} s_1 &= t_1, \\ s_2 &= t_3, \\ s_3 &= t_6, \\ &\vdots \\ s_n &= t_{n(n-1)/2}. \end{aligned}$$

Therefore if $(t_n)_{n=1}^{\infty}$ converges, then $(s_n)_{n=1}^{\infty}$, being a subsequence, also converges to the same limit.

(b) Rearrange the terms of $\sum_{n=1}^{\infty} x_n$ by writing the sum as

$$\left(\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} \ldots\right) + \left(\tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \ldots\right) + \left(\tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \ldots\right) + \ldots$$

(This part is more of a "thinking question", requiring you to think about why this rearrangement makes sense and why it is valid to perform it).

(c) Evaluate each of the internal sums in the above rearrangement, and the sum of their values, to conclude that $S = 2$.

Solution. This was explained in class in the last lecture.