Homework Set No. 2 — MAT 280, Fall 2013

Due: 11/14/13

1. (a) Let $q(n)$ denote the number of partitions of an integer n into *odd* parts. Explain the generating function identity

$$1 + \sum_{n=1}^{\infty} q(n) x^n = \prod_{m=1}^{\infty} \frac{1}{1 - x^{2m-1}}.$$

(There is no need to discuss issues of convergence.)

(b) Let $r(n)$ denote the number of partitions of an integer n into *distinct* parts (i.e., with no repetitions allowed). Find an infinite product formula for the generating function of $r(n)$, of the form

$$1 + \sum_{n=1}^{\infty} r(n) x^n = \prod_{m=1}^{\infty} [\ ???\ ].$$

(c) Prove that $q(n) = r(n)$ for all $n \geq 1$.

2. Define partition-counting functions $A(n), B(n), C(n)$ by

$$\begin{aligned} A(n) &= \#\text{ of partitions of } n \text{ with no 1's and no two consecutive parts}, \\ B(n) &= \#\text{ of partitions of } n \text{ with no part appearing exactly once}, \\ C(n) &= \#\text{ of partitions of } n \text{ with no part } k \text{ satisfying } k \equiv \pm 1 \pmod 6). \end{aligned}$$

Prove that $A(n) = B(n) = C(n)$ for all $n \geq 1$.

Hints. First, to make sure you understand the definitions, it may be a good idea to start by verifying this directly for $n = 2, 3, 4$. Second, prove separately that $A(n) = B(n)$ using a simple graphical observation about Young diagrams, and that $B(n) = C(n)$ using generating functions.

3. Let $m, n, k \geq 1$ be integers. A **generalized permutation of length** k **and row bounds** (m, n) is a two-line array of integers which has the form

$$\sigma = \begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix},$$

where $1 \le i_1, \ldots, i_k \le m,\ 1 \le j_1, \ldots, j_k \le n$, and where the columns are ordered lexicographically, in the sense that if $s < t$ then either $i_s < i_t$, or $i_s = i_t$ and $j_s \le j_t$. Denote by $\mathcal{P}^k_{m,n}$ the set of generalized permutations of length k and row bounds (m, n).

Next, let $\mathcal{M}^k_{m,n}$ denote the set of $m \times n$ matrices $(a_{i,j})_{1\le i\le m, 1\le j\le n}$ with nonnegative integer entries satisfying $\sum_{i,j} a_{i,j} = k$. For each generalized permutation $\sigma \in \mathcal{P}^k_{m,n}$ define a matrix $M_\sigma = (a_{i,j})_{i,j} \in \mathcal{M}^k_{m,n}$ by setting $a_{i,j}$ to be the number of columns in σ equal to $\binom{i}{j}$.

(a) Explain why the mapping $\sigma \mapsto M_\sigma$ establishes a bijection between $\mathcal{P}^k_{n,m}$ and $\mathcal{M}^k_{m,n}$.

(b) Find M_σ when

$$\sigma = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\ 1 & 1 & 4 & 5 & 3 & 3 & 3 & 5 & 2 & 5 \end{pmatrix}$$

(considered as an element of $\mathcal{P}^{10}_{3,5}$).

(c) Find σ when

$$M_\sigma = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 & 3 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 0 \end{pmatrix}.$$

(d) Find a formula for the number $|\mathcal{P}^k_{m,n}|$ of generalized permutations of length k with row bounds (m, n).

(e) If $\sigma = \left(\begin{smallmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{smallmatrix}\right)$ is a generalized permutation and $1 \le s_1 < \ldots < s_d \le k$ is a sequence of column positions, we refer to the generalized permutation $\left(\begin{smallmatrix} i_{s_1} & \cdots & i_{s_d} \\ j_{s_1} & \cdots & j_{s_d} \end{smallmatrix}\right)$ as a **subsequence of** σ, and call such a subsequence **increasing** if $j_{s_1} \le \ldots \le j_{s_d}$. If $\sigma \in \mathcal{P}^k_{m,n}$, as for ordinary permutations let $L(\sigma)$ denote the maximal length of an increasing subsequence of σ. Equivalently, $L(\sigma)$ is the maximal length of a weakly increasing subsequence of the bottom row of σ.

Given a matrix $M = (a_{i,j})_{i,j} \in \mathcal{M}^k_{m,n}$, define

$$G(M) = \max\left\{ \sum_{\ell=0}^{r} a_{i_\ell, j_\ell} \,:\, (i_0, j_0) \to (i_1, j_1) \to \ldots \to (i_r, j_r) \text{ is an up-right path in } \mathcal{Z}(1,1;m,n) \right\}.$$

(That is, the definition of $G(M)$ as a function of the entries $a_{i,j}$ of M is the same as the definition of the passage time $G(m, n)$ in terms of the clock times $(\tau_{i,j})_{1\le i\le m, 1\le j\le n}$; see page 254 in the book.) Prove that if $\sigma \in \mathcal{P}^k_{m,n}$ and $M = M_\sigma$ is the associated matrix in $\mathcal{M}^k_{m,n}$ then

$$G(M) = L(\sigma).$$

Hints. To prove that $G(M) \le L(\sigma)$, show how one can associate to any up-right path $(i_0, j_0) \to \ldots \to (i_r, j_r)$ in $\mathcal{Z}(1, 1; m, n)$ a subsequence of σ of length $\sum_{\ell=0}^{r} a_{i_\ell, j_\ell}$. (To get some intuition, it may be useful to try this first with a concrete example such as the ones in parts (b), (c) above.)

On the other hand, for an increasing subsequence of σ of length k, by considering the distinct columns $\binom{i}{j}$ appearing in an increasing subsequence of σ of length k, show that $k \le \sum_{\ell=0}^{r} a_{i_\ell, j_\ell}$ for a suitable up-right path, and deduce that $L(\sigma) \le G(M)$.

4. Solve problem 4.9 in the book (pages 300–301).

Note. Problems 4.10(a),(b) will be part of the next homework set, so you can start working on that if you have time.