Homework Set No. 2 — MAT 280, Fall 2013

Due: 11/14/13

1.  (a) Let \( q(n) \) denote the number of partitions of an integer \( n \) into odd parts. Explain the generating function identity

\[
1 + \sum_{n=1}^{\infty} q(n)x^n = \prod_{m=1}^{\infty} \frac{1}{1 - x^{2m-1}}.
\]

(There is no need to discuss issues of convergence.)

(b) Let \( r(n) \) denote the number of partitions of an integer \( n \) into distinct parts (i.e., with no repetitions allowed). Find an infinite product formula for the generating function of \( r(n) \), of the form

\[
1 + \sum_{n=1}^{\infty} r(n)x^n = \prod_{m=1}^{\infty} \left[ ??? \right].
\]

(c) Prove that \( q(n) = r(n) \) for all \( n \geq 1 \).

2. Define partition-counting functions \( A(n), B(n), C(n) \) by

\[
A(n) = \# \text{ of partitions of } n \text{ with no 1’s and no two consecutive parts},
\]

\[
B(n) = \# \text{ of partitions of } n \text{ with no part appearing exactly once},
\]

\[
C(n) = \# \text{ of partitions of } n \text{ with no part } k \text{ satisfying } k \equiv \pm 1 \pmod{6}.
\]

Prove that \( A(n) = B(n) = C(n) \) for all \( n \geq 1 \).

Hints. First, to make sure you understand the definitions, it may be a good idea to start by verifying this directly for \( n = 2, 3, 4 \). Second, prove separately that \( A(n) = B(n) \) using a simple graphical observation about Young diagrams, and that \( B(n) = C(n) \) using generating functions.

3. Let \( m, n, k \geq 1 \) be integers. A generalized permutation of length \( k \) and row bounds \( (m,n) \) is a two-line array of integers which has the form

\[
\sigma = \begin{pmatrix}
i_1 & i_2 & \cdots & i_k \\
j_1 & j_2 & \cdots & j_k
\end{pmatrix},
\]

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where $1 \leq i_1, \ldots, i_k \leq m$, $1 \leq j_1, \ldots, j_k \leq n$, and where the columns are ordered lexicographically, in the sense that if $s < t$ then either $i_s < i_t$, or $i_s = i_t$ and $j_s \leq j_t$. Denote by $\mathcal{P}_{m,n}^k$ the set of generalized permutations of length $k$ and row bounds $(m,n)$.

Next, let $\mathcal{M}_{m,n}^k$ denote the set of $m \times n$ matrices $(a_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}$ with nonnegative integer entries satisfying $\sum_{i,j} a_{i,j} = k$. For each generalized permutation $\sigma \in \mathcal{P}_{m,n}^k$ define a matrix $M_\sigma = (a_{i,j})_{i,j} \in \mathcal{M}_{m,n}^k$ by setting $a_{i,j}$ to be the number of columns in $\sigma$ equal to $(i_j)$.

(a) Explain why the mapping $\sigma \mapsto M_\sigma$ establishes a bijection between $\mathcal{P}_{m,n}^k$ and $\mathcal{M}_{m,n}^k$.

(b) Find $M_\sigma$ when 

$$\sigma = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ 1 & 1 & 4 & 5 & 3 & 3 & 5 & 2 & 5 \end{pmatrix}$$

(considered as an element of $\mathcal{P}_{3,5}^{10}$).

(c) Find $\sigma$ when 

$$M_\sigma = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 & 3 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 0 \end{pmatrix}.$$  

(d) Find a formula for the number $|\mathcal{P}_{m,n}^k|$ of generalized permutations of length $k$ with row bounds $(m,n)$.

(e) If $\sigma = \begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix}$ is a generalized permutation and $1 \leq s_1 < \ldots < s_d \leq k$ is a sequence of column positions, we refer to the generalized permutation $\begin{pmatrix} i_{s_1} & \cdots & i_{s_d} \\ j_{s_1} & \cdots & j_{s_d} \end{pmatrix}$ as a subsequence of $\sigma$, and call such a subsequence increasing if $j_{s_1} \leq \ldots \leq j_{s_d}$. If $\sigma \in \mathcal{P}_{m,n}^k$, as for ordinary permutations let $L(\sigma)$ denote the maximal length of an increasing subsequence of $\sigma$. Equivalently, $L(\sigma)$ is the maximal length of a weakly increasing subsequence of the bottom row of $\sigma$.

Given a matrix $M = (a_{i,j})_{i,j} \in \mathcal{M}_{m,n}^k$, define 

$$G(M) = \max \left\{ \sum_{t=0}^{r} a_{i_t,j_t} : (i_0,j_0) \to (i_1,j_1) \to \ldots \to (i_r,j_r) \text{ is an up-right path in } Z(1,1;m,n) \right\}.$$
(That is, the definition of $G(M)$ as a function of the entries $a_{i,j}$ of $M$ is the same as the
definition of the passage time $G(m,n)$ in terms of the clock times $(τ_{i,j})_{1≤i≤m,1≤j≤n}$; see page
254 in the book.) Prove that if $σ ∈ P_{m,n}^k$ and $M = M_σ$ is the associated matrix in $M_{m,n}^k$
then

\[ G(M) = L(σ). \]

Hints. To prove that $G(M) ≤ L(σ)$, show how one can associate to any up-right path
$(i_0,j_0) → \ldots → (i_r,j_r)$ in $Z(1,1;m,n)$ a subsequence of $σ$ of length $∑_{ℓ=0}^ra_{i_ℓ,j_ℓ}$. (To get
some intuition, it may be useful to try this first with a concrete example such as the ones in
parts (b), (c) above.)

On the other hand, for an increasing subsequence of $σ$ of length $k$, by considering the distinct
columns $\begin{pmatrix}i \\ j \end{pmatrix}$ appearing in an increasing subsequence of $σ$ of length $k$, show that $k ≤ ∑_{ℓ=0}^ra_{i_ℓ,j_ℓ}$
for a suitable up-right path, and deduce that $L(σ) ≤ G(M)$.


Note. Problems 4.10(a),(b) will be part of the next homework set, so you can start working
on that if you have time.