Homework Set No. 3 — MAT 280, Fall 2013

Due: 12/5/13

1. Given a convex function $f : \mathbb{R} \to \mathbb{R}$, let f^* denote its Legendre transform, defined by

$$f^*(p) = \max\{px - f(x) \,:\, x \in \mathbb{R}\}.$$

In this problem we assume that f is smooth and strictly convex, i.e., satisfies $f'' > 0$.

(a) Show that f^* is convex.

(b) Show that the Legendre transform is its own inverse, i.e., that $(f^*)^*(x) = f(x)$.

Hint for (a) and (b). Use the representation in equation (4.45) (page 274 in the notes) of the Legendre transform in the case of a smooth function.

(c) (Reading assignment—no submission required) Read the proof that $(f^*)^* = f$ in the general case that $f : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is a convex, but not necessarily smooth, function, as explained in Appendix C.1 (pages 93–94) of *Lecture Notes on the Corner Growth Model* by Timo Seppäläinen, available at this URL:
`http://www.math.wisc.edu/~seppalai/cornergrowth-book/ajo.pdf`.

2. Compute the Legendre transforms of the following functions:

(a) $f(x) = x^\alpha, \quad \alpha > 1$.

(b) $f(x) = e^x$.

(c) $f(x) = \cosh x$.

3. Solve problems 4.10(a),(b) in the lecture notes (pages 301–302). When working on part (b), you may replace the claim about $\phi_n(x) = n^{-1/2}\Lambda^{(n)}_{\lfloor n^{1/2}x+1\rfloor}$ (on the bottom of page 301) with a similar claim about the conjugate partition, i.e., redefine $\phi_n(x)$ as

$$\phi_n(x) = n^{-1/2}\Gamma^{(n)}_{\lfloor n^{1/2}x+1\rfloor}$$

where $\Gamma^{(n)}$ is the conjugate partition of $\Lambda^{(n)}$ (this result is slightly easier to prove—see the hints below—but it is not hard to see that both claims are equivalent).

Hints. For part (a), you may use the method described in notes I wrote on the Hardy-Ramanujan formula, which can be found here:

`https://www.math.ucdavis.edu/~romik/downloads/hardy-ramanujan.pdf`

For part (b), note that the kth part of $\Gamma^{(n)}$ can be represented as

$$\Gamma_k^{(n)} = \text{the number of parts of } \Lambda^{(n)} \text{ that are greater than } k = \sum_{j=k}^{\infty} N_j$$

(where N_j are defined at the beginning of exercise 4.9 on page 300) and generalize the technique of part (a) to get the result. For more related ideas see the paper *Statistical mechanics of combinatorial partitions and their limit shapes* by A. Vershik, which can be found here:

`http://www.pdmi.ras.ru/~vershik/statph.ps`

(for Russian-speakers, the original Russian version of this paper may be more helpful and can probably be found by googling).