Homework Set 1

Homework due: Wednesday 2/2/11

1. Let $\Lambda = \lim_{n\to\infty} \ell_n/\sqrt{n}$ as in Theorem 6 in the lecture notes. The goal of this problem is to show that the bounds $1 \le \Lambda \le e$ that follow from Lemmas 3 and 4 can be improved to $(8/\pi)^{1/2} \le \Lambda \le 2.49$.

   (a) In the proof of Lemma 4 observe that if $L(\sigma_n) \ge t$ then $X_{n,k} \ge \binom{t}{k}$, so the bound in (1.4) in the notes can be improved. Take $k \approx \alpha\sqrt{n}$ and $t \approx \beta\sqrt{n}$ and optimize the improved bound over $\alpha < \beta$ (using some version of Stirling's formula) to conclude that $\Lambda \le 2.49$.

   (b) Given a standard Poisson Point Process (PPP) Π in $[0,\infty)\times[0,\infty)$, construct an increasing subsequence $(X_1, Y_1), (X_2, Y_2), (X_3, Y_3), \ldots$ of points from the process by letting (X_1, Y_1) be the Poisson point that minimizes the coordinate sum $x + y$, and then by inductively letting (X_k, Y_k) be the Poisson point in $(X_{k-1},\infty)\times(Y_{k-1},\infty)$ that minimizes the coordinate sum $x + y$. Observe that the properties of the Poisson point process imply that one can write

   $$(X_k, Y_k) = \sum_{j=1}^{k} (W_j, Z_j),$$

   where $\big((W_j, Z_j)\big)_{j=1}^{\infty}$ is a sequence of independent and identically distributed random vectors in $[0,\infty)^2$, each having the same distribution as (X_1, Y_1). Compute the expectations $\mu_X = \mathbb{E}(X_1), \mu_Y = \mathbb{E}(Y_1)$.

   Hint: First, compute the density function of $X_1 + Y_1$. The only thing that one needs to know about the Poisson point process is that the probability that a measurable set $A \subset [0,\infty)^2$ does not contain any points of Π is equal to $\exp(-|A|)$ (where $|A|$ is the Lebesgue measure of A). Next, observe that conditioned on the event $X_1 + Y_1 = t$, the conditional distribution of X_1 is uniform in $[0, t]$. This follows from one of the properties of the PPP, namely that conditioned on the event that a measurable set A contains k points of Π, the distribution of the location of these points is simply that of k independently chosen uniformly random points in A; the observation follows by applying this property with $k = 1$ and the set $A = \{(x, y) : t \le x + y \le t + \Delta t\}$ and letting $\Delta \to 0$.

(c) Now apply the Strong Law of Large Numbers to the sequences X_k and Y_k, to get that $(X_k, Y_k)/k \to (\mu_X, \mu_Y)$ almost surely as $k \to \infty$. On the other hand, use the fact (used in the proof of Hammersley's theorem on the existence of the limit $\Lambda = \lim_{n\to\infty} \ell_n/\sqrt{n}$) that Λ can also be expressed in terms of the Poisson point process as the almost-sure limit

$$\Lambda = \lim_{n\to\infty} L(\Pi \cap [0, \sqrt{n}]^2)$$

(where $L(\cdot)$ denotes the maximal increasing subsequence length) to conclude that $\Lambda \ge (8/\pi)^{1/2} \approx 1.596$.

2. Let $p(n)$ denote the number of partitions of an integer n (or equivalently the number of Young diagrams of order n). Show that there exists a constant $c > 0$ such that the inequality $p(n) > e^{c\sqrt{n}}$ holds for all $n \ge 1$.

3. For $n \in \mathbb{N}$, let $c(n)$ denote the number of **ordered partitions** (also called **compositions**) of n, i.e., ways of expressing n as a sum of positive integers, where different orders are considered distinct representations. Prove that $c(n) = 2^{n-1}$, and deduce trivially that $p(n) \le 2^{n-1}$.

4. (a) Define the generating function $F(z) = 1 + \sum_{n=1}^{\infty} p(n)z^n$. It is traditional to define $p(0) = 1$, so this can also be written as $F(z) = \sum_{n=0}^{\infty} p(n)z^n$. Deduce from the previous problem that the series converges absolutely and uniformly on compacts in the region $|z| < 1/2$. Prove that in this range **Euler's product formula** holds:

$$F(z) = \prod_{k=1}^{\infty} \frac{1}{1 - z^k}.$$

(**Hint:** Expand each of the factors in the infinite product as a geometric series, and interpret the coefficient of z^n in the resulting power series combinatorially.)

(b) Show that the product on the right-hand side of Euler's product formula actually converges absolutely and uniformly on compacts, and therefore defines an analytic function, in the region $\{z \in \mathbb{C} : |z| < 1\}$. Deduce that the power series defining $F(z)$ also converges absolutely in this region.

(c) Show that if $0 < x < 1$ then $F(x) \le \frac{\pi^2}{6} \frac{x}{1-x}$. (You may need to use the fact that $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$).

(d) Show that for real x satisfying $0 < x < 1$ we have $p(n) < x^{-n}F(x)$. Using the bound above for $F(x)$, find a value of x (as a function of n) that makes this a particularly good bound, and deduce that the bound $p(n) \leq e^{\pi\sqrt{2n/3}}$ holds for all $n \geq 1$.

(**Hint:** look for an x of the form $x = x_n = 1 - \frac{\alpha}{\sqrt{n}}$ for a suitable constant $\alpha > 0$.)

5. Let $\sigma = (8, 2, 10, 9, 3, 1, 5, 4, 7, 6) \in S_{10}$. Compute the triple (λ, P, Q) corresponding to σ via the Robinson-Schensted algorithm. Compute the permutation σ' obtained by applying the *inverse* Robinson-Schensted algorithm to the transposed Young diagram and pair of standard Young tableaux $(\lambda', P^\top, Q^\top)$. If you wish to get more practice, repeat the computation starting with the permutation $\sigma = (16, 8, 11, 13, 9, 2, 5, 6, 14, 7, 3, 12, 4, 1, 15, 10) \in S_{16}$.

6. a. Show that the number of involutions (self-inverse permutations) in S_n, and hence also the number of standard Young tableaux of order n, is equal to

$$I_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^k k!(n-2k)!}.$$

(**Hint:** consider the cycle structure of an involution.)

b. (**Optional**) Show that I_n can be expressed as

$$I_n = \mathbb{E}(Z^n),$$

where Z is a random variable with distribution $N(1, 1)$ (normal distribution with mean 1 and variance 1).

7. If λ is a Young diagram with m rows and row lengths

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$$

(where λ_j denotes the length of the j-th row if $j \leq m$, or 0 if $j > m$), show that for any $k \geq m$,

$$d_\lambda = |\lambda|! \frac{\prod_{1 \leq i < j \leq k}(\lambda_i - \lambda_j + j - i)}{\prod_{1 \leq i \leq k}(\lambda_i + k - i)!}.$$

Hint: First show that it is enough to prove this for $k = m$. Then prove that the identity

$$\prod_{j=1}^{\lambda_i} h_\lambda(i, j) = \frac{(\lambda_i + m - i)!}{\prod_{i < j \leq m}(\lambda_i - \lambda_j + j - i)}$$

holds for any $i = 1, \ldots, m$ (where $h_\lambda(i, j)$ denotes the hook length of the cell (i, j) in λ), and apply the hook length formula.

8. Recommended additional reading: Section 5.1.4 in Donald Knuth's *The Art of Computer Programming, Vol. 3: Sorting and Searching, 2nd Ed.* — a highly readable account of the Robinson-Schensted algorithm and some of its basic properties.