MAT 280 — UC Davis, Winter 2011
Longest increasing subsequences and combinatorial probability

# Homework Set 2

**Homework due: Wednesday 2/2/23**

1. If $f$ is a continual Young diagram and $g$ is its rotated version, let
$$K(g) = I_{\mathrm{hook}}(f) = \int_0^\infty \int_0^{f(x)} \log h_f(x,y)\,dy\,dx$$
be the hook integral that appears in the asymptotic version of the hook length formula
$$\frac{d_\lambda^2}{|\lambda|!} = \exp\left[-n\left(1 + 2I_{\mathrm{hook}}(\phi_\lambda) + O\left(\frac{\log n}{\sqrt{n}}\right)\right)\right].$$
We proved that the curve
$$\Omega(u) = \begin{cases} \frac{2}{\pi}\left(u \sin^{-1}\left(\frac{u}{\sqrt{2}}\right) + \sqrt{2-u^2}\right) & |u| \le \sqrt{2}, \\ |u| & |u| > \sqrt{2}. \end{cases}$$
minimizes $K(\cdot)$ among the (rotated) continual Young diagrams. Explain (as rigorously as you can) how this can be used to deduce in a computation-free way the evaluation
$$K(\Omega) = -\tfrac{1}{2}.$$

2. Prove the Cauchy determinant identity
$$\det\left(\frac{1}{x_i + y_j}\right)_{i,j=1}^n = \frac{\prod_{1\le i<j<n}(x_j - x_i)(y_j - y_i)}{\prod_{1\le i,j\le n}(x_i + y_j)}.$$
**Hint:** This determinant can be evaluated in many ways. One of the easiest involves performing a simple "elementary operation" on columns 2 through $n$ of the Cauchy matrix, extracting factors common to rows and columns and then performing another simple elementary operation on rows 2 through $n$.

3. Let $\lambda$ be a Young diagram, and let $(p_1, \ldots p_d \mid q_1, \ldots, q_d)$ denote its Frobenius coordinates, where
$$p_i = \lambda_i - i, \qquad q_i = \lambda_i' - i \qquad (i = 1, \ldots, d).$$

Show that the set of *modified* Frobenius coordinates

$$\left\{p_1 + \tfrac{1}{2}, \ldots, p_d + \tfrac{1}{2}, -q_1 - \tfrac{1}{2}, \ldots, -q_d - \tfrac{1}{2}\right\}$$

is equal to the set

$$\left\{\lambda_i - i + \tfrac{1}{2} \,:\, i = 1, 2, \ldots\right\} \triangle \left\{-\tfrac{1}{2}, -\tfrac{3}{2}, -\tfrac{5}{2}, \ldots\right\},$$

where $A \triangle B = (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference of two sets $A$ and $B$, and where we define $\lambda_i$ as 0 if $i$ is greater than the number of parts of $\lambda$.

**Hint:** The $j$-th column length of $\lambda$ is given by $\lambda'_j = \#\{i : \lambda_i \geq j\}$.

4. (a) If $M = (m_{i,j})_{i,j=1}^n$ is a square matrix and $A \subseteq \{1, \ldots, n\}$, we denote $\det_A(M) = \det\left(m_{i,j}\right)_{i,j \in A}$ (the minor of $M$ corresponding to the rows and columns indexed by elements of $A$). Prove the following identity due to Jacobi: If $M$ is an invertible $n \times n$ matrix and $A \subseteq \{1 \ldots, n\}$, $A^c = \{1, \ldots, n\} \setminus A$, then

$$\det_A(M) = \det(M) \det_{A^c}(M^{-1}).$$

**Hint:** Assume without loss of generality that $A = \{1, \ldots, k\}$ for some $0 \leq k \leq n$. Write $M$ and $M^{-1}$ as block matrices

$$M = \begin{pmatrix} B & C \\ D & E \end{pmatrix}, \qquad M^{-1} = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix},$$

where $B$ and $W$ are $k \times k$ blocks, and compute in two ways the determinant of the matrix product

$$\begin{pmatrix} B & C \\ D & E \end{pmatrix} \begin{pmatrix} I & X \\ 0 & Z \end{pmatrix}.$$

(b) Can you see a connection between this result and Propositions 9 and 10 in Chapter 2 of the lecture notes?

**Hint:** If $X$ is a determinantal point process on a finite set $\Omega$ with defining kernel $L : \Omega \times \Omega \to \mathbb{R}$, where $L$ is an invertible operator, consider the complementary process $X^c = \Omega \setminus X$.

5. The purpose of this problem is to introduce *Fristedt's model for random partitions*, a probabilistic construction introduced by Bert Fristedt that is useful for studying the behavior of uniformly random partitions. Fix a parameter

$0 < x < 1$. Let $X_1, X_2, \ldots$ be a family of independent random variables such that for each $m \geq 1$, $X_m$ has distribution $\mathrm{Geo}_0(x^m)$ (the geometric distribution starting from 0 with parameter $x^m$); i.e., we have

$$\mathbb{P}_x(X_m = k) = (1 - x^m)x^{mk}, \qquad (k \geq 0),$$

where the notation $\mathbb{P}_x(\cdot)$ is meant to remind us that all probabilities depend on the parameter $x$. The sequence $(X_1, X_2, \ldots)$ encodes a random integer partition $\lambda \in \mathcal{P}^*$ which has $X_1$ parts equal to 1, $X_2$ parts equal to 2, etc. The order of the partition is the random variable

$$N = \sum_{m=1}^{\infty} m X_m.$$

(a) Prove that $N < \infty$ almost surely. (**Hint:** Use the Borel-Cantelli lemma.)

(b) Prove that for any partition $\mu \in \mathcal{P}^*$,

$$\mathbb{P}_x(\lambda = \mu) = \frac{x^{|\mu|}}{F(x)},$$

where $F(x) = \prod_{m=1}^{\infty}(1 - x^m)^{-1}$. Deduce that

$$\mathbb{P}_x(N = n) = \frac{p(n)x^n}{F(x)}, \qquad (n \geq 0)$$

and that, conditioned on the event $N = n$, the random partition $\lambda$ is distributed uniformly on the set of partitions of order $n$.

(c) By considering the fact that $1 = \sum_{n=0}^{\infty} \mathbb{P}_x(N = n)$ (that follows from the result of part (a)), deduce a new proof of the identity

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{m=1}^{\infty} \frac{1}{1 - x^n}.$$

(d) Set the parameter $x$ to be $x = x_n = \exp(-\frac{\pi}{\sqrt{6n}})$ (note that $x$ depends on $n$ — the idea is that this choice helps us understand the asymptotic behavior of uniformly random partitions of order $n$, when $n$ is large). For this choice of $x$, show that asymptotically as $n \to \infty$, we have

$$\mathbb{E}_x(N) = (1 + O(1/\sqrt{n}))n,$$
$$\mathrm{Var}_x(N) = (1 + o(1))\frac{2\sqrt{6}}{\pi} n^{3/2}.$$

That is, the random partition $\lambda$ has order whose mean is approximately $n$ and whose typical deviation from the mean is of order $n^{3/4}$ (by Chebyshev's inequality).

(e) To be continued...