MAT 280 — UC Davis, Winter 2011
Longest increasing subsequences and combinatorial probability

Homework Set 3

Homework due: Wednesday 3/16/11

1. Prove the summation identity

$$\sum_{k=0}^{n} \frac{(-1)^k}{k+\alpha}\binom{n}{k} = \frac{n!}{\alpha(\alpha+1)\ldots(\alpha+n)} \qquad (n \geq 0),$$

which was used in the proof of the Bessel function identity (2.40) in the lecture notes.

Hint: First prove, and then use, the following "binomial inversion principle": if $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are sequences such that $a_n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}b_k$ for all n, then the symmetric equation $b_n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}a_k$ also holds for all n.

2. Prove Hadamard's inequality from elementary linear algebra: if A is an $n \times n$ matrix whose column vectors are $v_1, \ldots, v_n \in \mathbb{R}^n$, then

$$|\det(A)| \leq \prod_{j=1}^{n} ||v_j||_2.$$

What is the geometric meaning of this inequality?

3. Let Ω be a countable set. Let $\mathbf{M} : \Omega \times \Omega \to \mathbb{R}$ be a kernel which is symmetric (i.e., $\mathbf{M}(x,y) = \mathbf{M}(y,x)$ for all $x, y \in \Omega$) and positive-definite (i.e., if we think of $\mathbf{M}$ as a linear operator then $\langle \mathbf{Mx}, \mathbf{x}\rangle \geq 0$ for any vector $x \in \mathbb{R}^\Omega$ with finite support). The goal of this problem is to show that the Fredholm determinant $\det(\mathbf{I}+\mathbf{M})$ exists if and only if $\sum_{x\in\Omega} |\mathbf{M}(x,x)| < \infty$. Note that this also implies that if $\det(\mathbf{I}+\mathbf{M})$ exists then $\det(\mathbf{I}+z\mathbf{M})$ exists for all $z \in \mathbb{C}$ and defines an entire function of the complex variable z.

   (a) Prove that if $A = (a_{i,j})_{i,j=1}^n$ is a symmetric positive-definite matrix then $|\det A| \leq \prod_{i=1}^n a_{i,i}$. **Hint:** Such a matrix A can be written as $A = L^\top L$ for some matrix L. Apply Hadamard's inequality to L.

   (b) Prove that if $\det(\mathbf{I}+\mathbf{M})$ exists then $\sum_{x\in\Omega} |\mathbf{M}(x,x)| < \infty$.

(c) Prove that

$$\sum_{E\subset\Omega,\ |E|<\infty} |\det(\mathbf{M}_E)| \le \prod_{x\in\Omega} \big(1 + |\mathbf{M}(x,x)|\big),$$

and deduce that if $\sum_{x\in\Omega} |\mathbf{M}(x,x)| < \infty$ then $\det(\mathbf{I}+\mathbf{M})$ exists.

4. Let $J_\alpha(z)$ denote the Bessel function of the first kind of order α. Show that

$$\begin{aligned} J_{1/2}(z) &= \sqrt{\frac{2}{\pi z}}\sin z, \\ J_{-1/2}(z) &= \sqrt{\frac{2}{\pi z}}\cos z. \end{aligned}$$

5. Prove the following identities involving Bessel functions $J_n(z)$ of integer order:

$$\exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z) t^n, \tag{1}$$

$$\cos(z\sin\theta) = J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z)\cos(2n\theta), \tag{2}$$

$$\sin(z\sin\theta) = 2\sum_{n=0}^{\infty} J_{2n+1}(z)\sin((2n+1)\theta), \tag{3}$$

$$\cos(z\cos\theta) = J_0(z) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(z)\cos(2n\theta), \tag{4}$$

$$\sin(z\cos\theta) = 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(z)\sin((2n+1)\theta), \tag{5}$$

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos\left(z\sin\theta - n\theta\right)\, d\theta. \tag{6}$$

Hint for (6)**:** $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$.