1. (a) We rewrite the second and third equations in the system as

\[
(5 - z)x + 2y = 0 \\
2x + (8 - z)y = 0
\]

We are looking for a solution other than the zero solution \(x = y = 0\). By a previous exercise mentioned in the problem, such a solution exists if and only if

\[
(5 - z)(8 - z) - 2 \cdot 2 = 0
\]

This gives the quadratic equation

\[z^2 - 13z + 36 = 0,\]

which has the two solutions \(z = 4\) and \(z = 9\).

(b) When \(z = 4\), the solution of the original system of three equations is \(x = 2\), \(y = -1\). This doesn’t make sense since in the original problem \(x\) and \(y\) referred to the relative importances of two websites, and therefore had to be positive numbers (or 0).

When \(z = 9\), the solution is \(x = \frac{1}{3}\), \(y = \frac{2}{3}\), which makes sense and is the same solution given in the lecture notes.

2. (a) \(f(x) = x^3 - x^2 + 2\), \(g(x) = x - 2 \implies q(x) = x^2 + x + 2, r(x) = 6\)
(b) \(f(x) = x^3 - x^2 + 2\), \(g(x) = x + 1 \implies q(x) = x^2 - 2x + 2, r(x) = 0\)
(c) \(f(x) = x^4 + x\), \(g(x) = x^2 + 1 \implies q(x) = x^2 - 1, r(x) = x + 1\)

3. If \(p(x) = x^3 - 4x^2 + 2x + 3\) and \(p(3) = 0\), then \(p(x)\) is divisible (without remainder) by the factor \(x - 3\). By performing polynomial long division we find that

\[p(x) = (x - 3)(x^2 - x - 1)\]

It follows that the roots of \(p(x)\) are \(x_1 = 3\) and the two roots of \(x^2 - x - 1\), which are

\[x_2 = \frac{1 - \sqrt{5}}{2}, \quad x_3 = \frac{1 + \sqrt{5}}{2}\]

4. \[x^n - 1 = (x - 1)(1 + x + x^2 + \ldots + x^{n-1})\]

5. Solve the following problems in the textbook:

(a) Calculational exercise 1 in Chapter 3.

**Solution to both parts (a) and (b).** The polynomial \(p(x) = x\) satisfies the condition (it is of degree 1 which is \(\leq 2\)).

(b) Proof-writing exercise 2(a) in Chapter 3.

**Solution to part (a).** First, assume that \(p(z)\) is just a single monomial of the form \(a_kz^k\) for some power \(k\). In this case, the equation \(\overline{p(z)} = p(\overline{z})\) which we need to prove becomes

\[\overline{(a_kz^k)} = a_k\overline{z}^k.\]

This is true, and follows from the property \(\overline{wz} = \overline{w} \cdot \overline{z}\) (“the conjugate element of a product is the product of the conjugate elements”) which holds for any two complex numbers.
Next, if the claim is true separately for each of two polynomials $p_1(z)$ and $p_2(z)$, then it is true for their sum $p(z) = p_1(z) + p_2(z)$, since

$$p(z) = p_1(z) + p_2(z) = \overline{p_1(z)} + \overline{p_2(z)} = \overline{p_1(z) + p_2(z)} = \overline{p(z)}$$

(here, we use the fact that the conjugate element of a sum of numbers is the sum of the conjugate elements).

By combining the two observations above (1. the claim is true for a monomial; 2. if it is true for two polynomials then it is true for their sum), we deduce that the claim is true also for a sum of monomials, so it is true for a general polynomial.

(c) Calculational exercise 1 in Chapter 4.

Solution.

(a) It is a vector space.

(b) It is a vector space.

(c) It is not a vector space. For example, the vectors $(0, 1)$ and $(1, 1)$ are both in the set but their sum $(0, 2)$ is not in the set. Thus, this subset of the vector space $\mathbb{R}^2$ is not closed under addition.

(d) It is not a vector space. For example, the vector $(1, 0)$ is in the set but multiplying it by the scalar $-1$ gives $(-1, 0)$ which is not in the set. So, the set is not closed under scalar multiplication.

(e) It is not a vector space, for similar reasons as the previous two examples.

(f) It is a vector space.

(g) It is not a vector space since it is not closed under addition (the “1” in the corner makes all the difference...)

Notes: To prove that a given set is a vector space, you need to demonstrate that all the properties in the definition of a vector space are satisfied. If the set is given as a subset of another set already known to be a vector space, it is enough to check just three properties (that it contains the zero vector, and is closed under addition and multiplication by a scalar).

On the other hand, to prove that a set is not a vector space, it is enough to show that one of the properties in the definition (or one of the three subspace properties, for a subset of a vector space) is not satisfied.

(d) Proof-writing exercise 1 in Chapter 4.

Solution. Assume that $a \in \mathbb{F}$, $v \in V$ satisfy $av = 0$. We claim that either $a = 0$ or $v = 0$. If $a = 0$, we are done. Otherwise, we show that $v$ must be the zero vector. Since $a \neq 0$, we can multiply the equation $av = 0$ by the reciprocal scalar $a^{-1}$. The left-hand side becomes

$$a^{-1}(av) = (a^{-1}a)v = 1 \cdot v = v$$

(the first equality follows from one of the “associativity” properties of a vector space; the second one follows from the “multiplicative identity” property). The right-hand side becomes $a^{-1}0$, which is also equal to the vector $0$ (see Proposition 4.2.4 in the textbook). The fact that the two expressions are equal gives the desired claim that $v = 0$. 