1. Solve the following problems in the textbook:

(a) Proof-writing exercise 6 in Chapter 5.
   Solution. Since $U + V \subset \mathbb{R}^9$, we have $\dim(U + V) \leq 9$, and therefore, by theorem 5.4.6 in the textbook,
   
   $$\dim(U \cap V) = \dim(U) + \dim(V) - \dim(U + V) \geq \dim(U) + \dim(V) - 9 = 5+5-9 = 1.$$ 
   
   So $U \cap V$ must be larger than the 0-dimensional space $\{0\}$.

(b) Calculational exercises 1(a),(b),(c),(f), 2(a)–(b), 5, 6 in Chapter 6.
   Solution to 1(b),(c). For each $(a, b) \in \mathbb{R}^2$, it is easy to find that the equation $(x+y, x) = T(x, y) = (a, b)$ has the unique solution $(x, y) = (b, a-b)$. Since there is a solution, that means the transformation is surjective. Since the solution is unique, that means the transformation is injective, which as we saw is equivalent to $\text{null}(T) = \{0\}$, so $\dim(\text{null}(T)) = 0$.

Solution to 1(f). $F(0,0) = (0,1)$. Since $F$ does not map the zero vector in the domain to the zero vector in the codomain, it is not a linear transformation.

(c) Proof-writing exercises 2, 3 in Chapter 6.
   Solution to 2. If $a_1, \ldots, a_n$ are scalars such that
   
   $$a_1T(v_1) + a_2T(v_2) + \ldots + a_nT(v_n) = 0,$$
   
   then, since $T$ is linear, we can rewrite this equation as
   
   $$T(a_1v_1 + \ldots + a_nv_n) = 0.$$
   
   Since we assumed $T$ is injective, its null space only contains the zero vector, so we get that
   
   $$a_1v_1 + \ldots + a_nv_n = 0$$
   
   (here, the right-hand side is the zero vector in $V$). We assumed the vectors $v_1, \ldots, v_n$ are linearly independent, so it must be that $a_1 = a_2 = \ldots = a_n = 0$. Thus, we showed that the only linear combination of $T(v_1), \ldots, T(v_n)$ that gives the zero vector is the trivial one with all coefficients equal to zero, therefore the vectors $T(v_1), \ldots, T(v_n)$ are linearly independent.

2. (a) Solve each of the following inhomogeneous linear systems.

   (i) \[
   \begin{aligned}
   x + 4z &= 2 \\
   y &= -1 
   \end{aligned}
   \]

   The solution set is $\{(2-4z,-1,z) : z \in \mathbb{R}\} = \{(2,-1,0) + z(-4,0,1) : z \in \mathbb{R}\}$

   (ii) \[
   \begin{aligned}
   3x - 3y + 15z &= -6 \\
   x + 2y - z - w &= -3 \\
   x + 3z + w &= -1 
   \end{aligned}
   \]
The solution set is \(\{(−2−3z, 2z, z, 1) : z ∈ \mathbb{R}\} = \{(−2, 0, 0, 1)+z(−3, 2, 1, 0) : z ∈ \mathbb{R}\}\)

(iii) \[
\begin{align*}
  x + 3y &= 1 \\
-x - 2y &= 3
\end{align*}
\]  \implies \text{unique solution} \ (x, y) = (−11, 4)

(iv) \[
\begin{align*}
  x + 3y &= 0 \\
3x + 9y &= 1
\end{align*}
\]  \implies \text{no solutions}

(v) \[
\begin{align*}
  x + y + 5z &= 13 \\
y - 10z &= 0
\end{align*}
\]  \implies \text{unique solution} \ (x, y, z) = (−17, 20, 2)

(vi) \[
\begin{align*}
  x_1 + 2x_2 + x_3 + 4x_4 + 4x_5 &= 5 \\
2x_1 + 4x_2 - x_3 + 5x_4 + 8x_5 &= −5 \\
x_1 + 2x_2 - 2x_3 + x_4 + 4x_5 &= −10 \\
x_1 + 2x_2 + 0x_3 + 6x_4 + 8x_5 &= 0
\end{align*}
\]  

The general form of the solution is \((x, y, z, s, t) = (−2y−3s−4t, y, 5−s, s, t), \ y, s, t ∈ \mathbb{R}\).

Alternatively, the solution set can be written as

\[
\{(0, 0, 5, 0, 0) + y(−2, 1, 0, 0, 0) + s(−3, 0, −1, 1, 0) + t(−4, 0, 0, 1, 0) : y, s, t ∈ \mathbb{R}\}
\]