1. Compute the coordinate vector \([v]_B\), where:

(a) \(v = (1, 0, 1), \ B = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}\).

Solution. \([v]_B = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\), since \(v = 1 \cdot (1, 0, 0) + (-1) \cdot (1, 1, 0) + 1 \cdot (1, 1, 1)\).

(b) \(v = (1, 0, 1), \ B = \{(1, 0, 1), (1, 0, 0), (0, 1, 0)\}\).

Solution. \([v]_B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\)

(c) \(v = z^3 - 2z, \ B = \{z + 1, z - 1, z^2, z^3\}\) in the space \(P_3\) of polynomials of degree \(\leq 3\).

Solution. \([v]_B = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}\)

2. Compute the representation matrix \(M(T)^B_C\), where:

(a) \(T(x, y) = (x + 10y, -x), \ B = C = \{(1, 0), (0, 1)\}\).

Solution. \(M(T)^B_C = \begin{pmatrix} 1 & 10 \\ -1 & 0 \end{pmatrix}\)

(b) \(T(x, y, z) = (z, y, 3x), \ B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}, \ C = \{(1, 0, 1), (0, 1, 0), (-1, 0, 1)\}\)

Solution. \(M(T)^B_C = \begin{pmatrix} 3/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 3/2 & 0 & -1/2 \end{pmatrix}\)

(c) \(T(x, y, z) = (z, y, 3x), \ B = \{(1, 0, 1), (0, 1, 0), (-1, 0, 1)\}, \ C = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}\)

Solution. \(M(T)^B_C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & -3 \end{pmatrix}\)

3. Solve calculational exercises 1(d), 1(e), 2(c), 3, 5 in Chapter 6.

Solution to 1(d). \(M(T) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\)

Solution to 1(e). \(M(T) = \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix}\)

Solution to 2(c). \(M(T) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\)

Solution to 3. Using Gaussian elimination we bring the matrix \(A\) to the reduced row-echelon form, which turns out to be \(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}\). This means that the solution set has the form

}\(null(S) = \{(0, -z, z) : z \in \mathbb{R}\} = \{z(0, -1, 1) : z \in \mathbb{R}\} = span\{(0, -1, 1)\}.\)
So, null(S) is 1-dimensional and has as a basis the single vector \((0, -1, 1)\).

**Solution to 5.** It is not difficult to see that the given space null\((T)\) has dimension equal to 2. By the dimension formula (Theorem 6.5.1 in the textbook), we have that

\[
4 = \dim(\mathbb{F}^4) = \dim(\text{null}(T)) + \dim(\text{range}(T)) = 2 + \dim(\text{range}(T)).
\]

So, we get that \(\dim(\text{range}(T)) = 2\). But \(\text{range}(T)\) is a subspace of the co-domain \(\mathbb{F}^2\), which is also two-dimensional. So, by a previous homework exercise (proof-writing exercise 4 in chapter 5), the two spaces are equal, i.e., \(\text{range}(T) = \mathbb{F}^2\), which means \(T\) is surjective.

4. Let \(U, V, W\) be finite-dimensional vector spaces, and let \(S : U \rightarrow V, T : V \rightarrow W\) be linear transformations. The goal of this problem is to prove the inequality:

\[
\dim(\text{null}(T \circ S)) \leq \dim(\text{null}(S)) + \dim(\text{null}(T)),
\]

where \(T \circ S : U \rightarrow W\) denotes the composition of the two transformations. Prove this by using the following steps:

(a) Denote \(H = \text{null}(T \circ S)\) (a linear subspace of \(U\)), and define a linear transformation \(R : H \rightarrow V\) by \(R(v) = S(v)\) (i.e., it is the same as \(S\), but its domain is a subspace of the domain of \(S\); sometimes \(R\) defined in this way will be referred to as the *restriction of \(S\) to \(H\)*). Show that \(\text{null}(S) \subseteq H\), and explain why this implies that \(\text{null}(R) = \text{null}(S)\).

**Solution.** If \(v \in \text{null}(S)\) then \(S(v) = 0\) and therefore also

\[
(T \circ S)(v) = T(S(v)) = T(0) = 0,
\]

so (by the definition of the null space) \(v \in \text{null}(T \circ S) = H\). For the second claim, remember that to prove two sets \(A\) and \(B\) are equal we need to prove that \(A \subseteq B\) and \(B \subseteq A\). In the present case, if \(v \in \text{null}(R)\) then \(v \in H\) and \(R(v) = S(v) = 0\), so also \(v \in \text{null}(S)\); this shows that \(\text{null}(R) \subseteq \text{null}(S)\). To show the reverse inclusion, \(\text{null}(S) \subseteq \text{null}(R)\), note that if \(v \in \text{null}(S)\) then, as we just showed, \(v \in H\), so \(R(v)\) is defined and equal to \(S(v) = 0\), therefore \(v \in \text{null}(R)\).

(b) Show that \(\text{range}(R) \subseteq \text{null}(T)\).

**Solution.** If \(u \in \text{range}(R)\) then there exists a vector \(v \in H\) (the domain of \(R\)) for which \(u = R(v) = S(v)\). It follows that

\[
T(u) = T(S(v)) = (T \circ S)(v) = 0,
\]

(since \(u \in H = \text{null}(T \circ S)\)). This shows that \(u \in \text{null}(T)\).

(c) Apply the dimension formula (Theorem 6.5.1 in the textbook) for a suitable linear transformation to deduce the inequality stated at the beginning of the question.

**Solution.** Applying the dimension formula to \(R : H \rightarrow U\) gives

\[
\dim(\text{null}(T \circ S) = \dim(H) = \dim(\text{null}(R)) + \dim(\text{range}(R)) = \dim(\text{null}(S)) + \dim(\text{range}(R)) \leq \dim(\text{null}(S)) + \dim(\text{null}(T)).
\]