1. (a) Compute the composition $\sigma \circ \pi$ of permutations, where:

i. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 6 & 2 & 4 & 5 \end{pmatrix}$, $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 1 & 2 & 3 & 4 \end{pmatrix}$

Solution.

$\sigma \circ \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 1 & 3 & 6 & 2 \end{pmatrix}$

ii. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{pmatrix}$, $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{pmatrix}$

Solution.

$\sigma \circ \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$

(b) Find the inverses of the permutations $\sigma$ and $\pi$ in part (a)-i. above.

Solution.

$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 5 & 6 & 3 \end{pmatrix}$, $\pi^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 2 & 1 \end{pmatrix}$

2. For each of the following matrices, compute its determinant and its adjoint matrix. If the matrix is invertible, use the adjoint matrix to find its inverse. (As usual, it is strongly recommended to check your answer by multiplying the matrix by the inverse matrix you found.)

(a) $\begin{pmatrix} 1 & 0 & -2 \\ 1 & 1 & 1 \\ 0 & -1 & 3 \end{pmatrix}$ $\implies$ $\det(A) = 1(1 \cdot 3 - 1(-1)) - 2(1 \cdot (-1) - 1 \cdot 0) = 6,$

$\text{adj}(A) = \begin{pmatrix} 4 & 2 & 2 \\ -3 & 3 & -3 \\ -1 & 1 & 1 \end{pmatrix}$, $A^{-1} = \frac{1}{6} \begin{pmatrix} 4 & 2 & 2 \\ -3 & 3 & -3 \\ -1 & 1 & 1 \end{pmatrix}$.

(b) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & -1 & 3 \end{pmatrix}$ $\implies$ $\det(A) = 1 \cdot 3 \cdot (2 \cdot (-5)(-1)) = 3,$

$\text{adj}(A) = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & 15 \\ 0 & 0 & 3 & 6 \end{pmatrix}$, $A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{pmatrix}$.

3. Let $A$ be a square matrix of order 4. We perform the following sequence of elementary row operations on $A$:

1. Subtract 3 times row 1 from row 2.
2. Multiply row 3 by $1/5$. 


3. Swap rows 1 and 4.
4. Add 10 times row 1 to row 2.
5. Swap rows 1 and 2.

After performing these operations, we get the new matrix
\[ B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}. \]

(a) Find a matrix \( D \) such that \( B = DA \).

**Hint:** Each of the elementary row operations can be represented as the operation of multiplying the current matrix from the left by an “elementary matrix”.

**Solution.** \( D \) is obtained by starting with the identity matrix (of order 4) \( I \) and performing the same sequence of elementary row operations. To see this, note that the relation between \( A \) and \( B \) can be encoded by the matrix multiplication equation
\[ B = E_5E_4E_3E_2E_1A = (E_5E_4E_3E_2E_1I)A = DA, \]
which explains the interpretation of the matrix \( D = E_5E_4E_3E_2E_1I \). With this knowledge, it is easy to compute and find that
\[ D = \begin{pmatrix} -3 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{5} & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \]

(b) Find \( \det(A) \).

**Solution.** \( \det(A) = \frac{\det(B)}{\det(E_1)\det(E_2)\det(E_3)\det(E_4)\det(E_5)} = \frac{6}{\frac{1}{5}(-1)(-1)} = 30. \)

4. Prove that for any \( n \), the number of permutations of order \( n \) with sign 1 is equal to the number of permutations with sign \(-1\).

**Solution.** Let \( t_{1,2} \) be the transposition of the numbers 1, \ldots, \( n \) that swaps the values 1 and 2 and leaves all other numbers unchanged. If we take a permutation \( \sigma \) with \( \text{sign}(\sigma) = 1 \) and define \( \pi = f(\sigma) = t_{1,2} \circ \sigma \), then we proved in class that
\[ \text{sign}(\pi) = -\text{sign}(\sigma) = -1. \]

Furthermore, the association \( \sigma \xrightarrow{f} \pi \) is invertible, since given \( \pi \) we can recover \( \sigma \) by noting that
\[ t_{1,2} \circ \pi = t_{1,2} \circ (t_{1,2} \circ \sigma) = (t_{1,2} \circ t_{1,2}) \circ \sigma = \text{id} \circ \sigma = \sigma \]
(the transposition \( t_{1,2} \) is an example of a permutation that it is its own inverse, i.e., \( t_{1,2} \circ t_{1,2} = \text{id} \)). Permutations which have this property are sometimes called *involutions*. Since an invertible function \( f : A \to B \) between two finite sets \( A \) and \( B \) can only exist if \( A \) and \( B \) have the same numbers of elements, the claim follows.

5. Solve the following exercises from the textbook:
(a) Calculational exercises 1 and 6 in Chapter 8.

Solution to 1. \( \det(A) = -2, \ \det(A^4) = (-2)^4 = 16. \)

Solution to 6. One can compute the determinant and show it is equal to 0; another approach is to observe that the third row of this matrix is equal to the sum of the first two rows. Thus, one can get a new matrix with a row of zeros (which is therefore not invertible) by performing elementary row operations (subtract the first and second rows from the third rows), which means that the original matrix is also not invertible.

(b) Proof-writing exercises 2 and 3 in Chapter 8.

Solution to 2. \( AA^\top \) is not invertible if and only if \( \det(AA^\top) = 0 \), but \( \det(AA^\top) = \det(A) \det(A^\top) = \det(A) \det(A) = \det(A)^2 \), and this is equal to 0 if and only if \( \det(A) = 0 \), which happens if and only if \( A \) is not invertible.

Solution to 3. Take \( A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \), \( B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \). Then \( A+B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \) so \( \det(A + B) = 1 \), but \( \det(A) = \det(B) = 0. \)