Problem 1

(a) Find the general form of the solution of the following system of linear equations:

\[
\begin{align*}
-x_1 + x_2 - 5x_3 + 2x_4 &= 0 \\
2x_1 - x_2 + 7x_3 - 24x_4 &= 0 \\
x_1 + x_2 - x_3 - 2x_4 &= 0 \\
-x_1 + 2x_2 - 8x_3 + 2x_4 &= 0 \\
\end{align*}
\]

Solution. Write the coefficient matrix and perform Gaussian elimination to get the matrix to reduced row-echelon form:

\[
\begin{pmatrix}
-1 & 1 & -5 & 2 \\
2 & -1 & 7 & -24 \\
1 & 1 & -1 & -2 \\
-1 & 2 & -8 & 2 \\
\end{pmatrix}
\]

\[
\xrightarrow{R_1 \leftarrow -R_1}
\xrightarrow{R_2 \leftarrow R_2 - 2R_1}
\xrightarrow{R_3 \leftarrow R_3 - R_1}
\xrightarrow{R_4 \leftarrow R_4 + R_1}
\xrightarrow{R_1 \leftarrow R_1 + R_2}
\xrightarrow{R_3 \leftarrow R_3 - 2R_1}
\xrightarrow{R_4 \leftarrow R_4 - R_2}
\xrightarrow{R_3 \leftarrow \frac{1}{2}R_3}
\xrightarrow{R_4 \leftarrow R_4 - 20R_3}
\begin{pmatrix}
1 & -1 & 5 & -2 \\
0 & 1 & -3 & -20 \\
0 & 2 & -6 & 0 \\
0 & 1 & -3 & 0 \\
\end{pmatrix}
\]

From the RREF we see that the solution set can be written as

\[
\{(−2x_3, 3x_3, x_3, 0) : x_3 \in \mathbb{R}\} = \{x_3(−2, 3, 1, 0) : x_3 \in \mathbb{R}\} = \text{span}\{(-2, 3, 1, 0)\}
\]

(b) If there is a solution except the trivial solution \(x_1 = x_2 = x_3 = x_4 = 0\), write explicitly one other solution of the system. That is, find some specific numbers \(x_1, x_2, x_3, x_4\), not all of them zero, which solve the system.

Solution. The vector \((-2, 3, 1, 0)\) (which is obtained from the general solution by setting \(x_3 = 1\)) is a nonzero solution.

(c) The set of solutions is a subspace of \(\mathbb{R}^4\). What is its dimension? Explain how you know.

Solution. The solution set was represented as the subspace spanned by the single vector \((-2, 3, 1, 0)\), so its dimension is 1 (the vector \{\((-2, 3, 1, 0)\)\} is a basis).
Problem 2

(a) For each of the following complex numbers $z$, compute the length $|z|:

1. $z = 3 + 4i$. Solution. $|z| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$.

2. $z = 10$. Solution. $|z| = \sqrt{10^2 + 0^2} = 10$.

3. $z = 2 + i$. Solution. $|z| = \sqrt{2^2 + 1^2} = \sqrt{5}$.

4. $z = 3(\cos(\pi/10) + \sin(\pi/10)i)$. Solution. $|z| = 3$ (z is given in polar coordinates).

5. $z = \frac{(1 - 2i)(1 + i)}{(3 + 4i)(1i)}$. Solution. $|z| = \frac{|1-2i||1+i|}{|3+4i|\|1i|} = \frac{\sqrt{5}\sqrt{2}}{\sqrt{5}\sqrt{2}} = \frac{\sqrt{10}}{\sqrt{5}}$.

(b) Prove that for any two complex numbers $z, w \in \mathbb{C}$, the parallelogram identity

$$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$$

holds.

Solution. We know that for any complex numbers $a, b$, $|a|^2 = a\overline{a}$, and $a + b = \overline{a} + \overline{b}$, so

$$|z + w|^2 = (z + w)(\overline{z + w}) = (z + w)(\overline{z} + \overline{w}) = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} = |z|^2 + z\overline{w} + w\overline{z} + |w|^2,$$

and similarly

$$|z - w|^2 = (z - w)(\overline{z - w}) = (z - w)(\overline{z} - \overline{w}) = z\overline{z} - z\overline{w} - w\overline{z} + w\overline{w} = |z|^2 - z\overline{w} - w\overline{z} + |w|^2.$$

Adding the two equations gives the result.

(c) Find all the solutions in complex numbers of the equation $z^3 + 3z = 0$.

Solution. The equation has one obvious solution $z_1 = 0$, so we can use this knowledge to factor the equation into the form

$$z(z^2 + 3) = 0,$$

which shows that the other two solutions are solutions to the equation $z^2 + 3 = 0$, or $z^2 = -3$, which are $z_2 = \sqrt{3}i$ and $z_3 = -\sqrt{3}i$. 
Problem 3

(a) For each of the following sets of vectors in a vector space, determine whether the set
is linearly independent, whether it spans the space, and whether it is a basis. In each
case explain why or why not.

1. \{ \mathbf{v}_1 = (1, -2, 0), \mathbf{v}_2 = (0, 0, 0), \mathbf{v}_3 = (1, 0, 0) \} in \mathbb{R}^3.

Solution. Since the set contains the zero vector, it is linearly dependent, for example
we have the linear combination

0 \cdot \mathbf{v}_1 + 1 \cdot \mathbf{v}_2 + 0 \cdot \mathbf{v}_3 = 0.

We know that \text{dim} \mathbb{R}^3 = 3. If the set were spanning, since it contains 3 vectors
it would also be a basis and hence linearly independent. But it is not linearly
independent, so it does not span \mathbb{R}^3, and is not a basis.

2. The polynomials \( p_1(z) = 1, p_2(z) = z, p_3(z) = z^2 + 1 \) in the space \( P_2 \) of polynomials
of degree \( \leq 2 \).

Solution. The polynomials span \( P_2 \), since a general polynomial \( p = a + bz + cz^2 \)
can be written as a linear combination

\[ p = ap_1 + bp_2 + c(p_3 - p_1) \]

They are also linearly independent (if we had \( ap_1 + bp_2 + cp_3 = 0 \), by comparing
coefficients this leads to the equations \( a + c = b = c = 0 \) which imply \( a = b = c = 0 \),
and are therefore a basis.

3. The vectors \( \mathbf{v}_1 = (1, 0, 1, 0), \mathbf{v}_2 = (1, 1, 0, 0), \mathbf{v}_3 = (1, 0, 0, 1) \) in \( \mathbb{R}^4 \).

Solution. The vectors cannot span \( \mathbb{R}^4 \) since there are only 3 of them and the
dimension of \( \mathbb{R}^4 \) is 4. They are linearly independent: the equation \( av_1 + bv_2 + cv_3 \)
leads immediately to the equations

\[ a = 0, b = 0, c = 0 \]

(and an extra equation \( a + b + c = 0 \), which is redundant).

(b) Recall that a linear transformation \( T : V \rightarrow W \) (where \( V,W \) are vector spaces) is called
\textit{injective} if for any vectors \( u, v \in V \), if \( u \neq v \) then \( T(u) \neq T(v) \); and \( T \) is called \textit{surjective}
if for any vector \( w \in W \) there exists a vector \( v \in V \) such that \( T(v) = w \).

For each of the following linear transformations, determine whether it is injective and
whether it is surjective. In each case explain why or why not it has either of these
properties.

1. \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \ T(x, y) = (x + y, x - y) \)

Solution. \( T \) is injective and surjective: the equation \( T(x, y) = (a, b) \) has a unique
solution \( x = (a + b)/2, y = (a - b)/2 \) for any \( a, b \).
2. $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(x, y, z) = (x + y, x - y)$

**Solution.** $T$ is not injective — the equation $T(x, y, z) = (0, 0)$ has a non-zero solution, for example $(x, y, z) = (0, 0, 1)$. But it is surjective since $T(x, y, z) = (a, b)$ if $(x, y, z) = ((a + b)/2, (a - b)/2, 0)$.

3. $T : \mathbb{R}^2 \to \mathbb{R}^3$, $T(x, y) = (x + y, x - y, x + 2y)$

**Solution.** $T$ is injective ($T(x, y) = (0, 0, 0)$ can only hold if $x = y = 0$), but not surjective: the equation $T(x, y) = (a, b, c)$ implies $x = (a + b)/2, y = (a - b)/2$ but for these values of $x, y$ it is not always true that $c = x + 2y$. That is, there are values of $(a, b, c)$ for which the equation cannot be solved, for example $a = 0, b = 0, c = 1$.

4. $T : V \to \mathbb{R}^2$, $T(x, y, z) = (x + y, y)$, where $V \subset \mathbb{R}^3$ is the set of solutions in $\mathbb{R}^3$ to the equation $z = 0$.

**Solution.** $T$ is injective and surjective: the equations $T(x, y, z) = (a, b)$, together with the requirement $z = 0$, always have the unique solution $x = a - b, y = b, z = 0$. 


Problem 4

(a) Let $U_1, U_2$ be two linear subspaces of a vector space $V$. Define the sum $U_1 + U_2$, and state when the sum $W = U_1 + U_2$ will be called a direct sum (in that case, we denote $W = U_1 \oplus U_2$).

(b) Prove that if $W = U_1 \oplus U_2$ then the only vectors $u_1 \in U_1, u_2 \in U_2$ satisfying $u_1 + u_2 = 0$ are $u_1 = u_2 = 0$.

(c) Prove that if $W = U_1 \oplus U_2$ then $U_1 \cap U_2 = \{0\}$.

Solutions. See the lecture of 10/7/11.